Classical Mechanics Derivations

Poisson's Equation

Solve for gravitational flux Φ_m

$$\Phi_m = \int_S n \cdot g \, da = -\int_S \frac{Gm\cos\theta}{r^2} da \underset{\text{using steradians}}{=} -4\pi G \int_V \rho \, dV$$

$$n \cdot g = -Gm \frac{\cos\theta}{r^2}$$

Gauss's Divergence Theorem:

$$\int_S n \cdot g \, da = \int_V \nabla \cdot g \, dV$$

$$\int_V -4\pi G \rho \, dV = \int_V \nabla \cdot g \, dV \quad \text{for arbitrary } V \text{ so} \quad -4\pi G\rho = \nabla \cdot g$$

substituting $g = -\nabla\Phi$

$$\boxed{\nabla^2 g = 4\pi G\rho}$$

Euler's Equation

Goal: Find extremum of $J = \int_{x_1}^{x_2} f\{y(x), y'(x); x\} dx = \int_{x_1}^{x_2} f\{y(\alpha,x), y'(\alpha,x); x\} dx$

$$y(\alpha,x) = y(0,x) + \alpha\eta(x)$$

Consider $\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{y'}\frac{\partial y'}{\partial \alpha} \right) dx$

$$\frac{\partial y}{\partial \alpha} = \eta(x) \quad \text{and} \quad \frac{\partial y'}{\partial \alpha} = \frac{d\eta}{dx}$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\frac{d\eta}{dx} \right) dx$$

Integration by parts:

$$\int u \, dv = uv - \int v \, du \qquad u = \frac{\partial f}{\partial y'} \qquad v = \eta(x) \, dx$$

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\frac{d\eta}{dx} dx = \frac{\partial f}{\partial y'} \cdot \eta(x) \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x) \, dx \cdot \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

$=0$

Since $\eta(x_2)=\eta(x_1)$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta(x) - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x) \right) dx = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right) \eta(x)\, dx$$

if $\frac{\partial J}{\partial \alpha} = 0$ $\quad \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$ must vanish since $\eta(x)$ is arbitrary

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0}$$

Euler's Equation w/ Constraints

Consider $f = f\{y, y', z, z'; x\}$ and $g\{y, z; x\} = 0$

$$dg = \left(\frac{\partial g}{\partial y} \cdot \frac{dy}{d\alpha} + \frac{\partial g}{\partial z} \cdot \frac{\partial z}{\partial \alpha} \right) d\alpha = 0$$

$\frac{\partial x}{\partial \alpha}$ vanishes

$$y(\alpha, x) = y(x) + \alpha \eta_1(x)$$

$$z(\alpha, x) = z(x) + \alpha \eta_2(x)$$

$$\frac{dy}{d\alpha} = \eta_1(x) \quad \text{and} \quad \frac{dz}{d\alpha} = \eta_2(x)$$

Plugging in to $\frac{dg}{d\alpha} = 0$

$$\frac{\partial g}{\partial y}\eta_1(x) = -\frac{\partial g}{\partial z}\eta_2(x)$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right)\eta_1(x) + \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'} \right)\eta_2(x) \right] dx$$

Factor out η_1 with $\frac{\eta_2(x)}{\eta_1(x)} = \frac{\frac{\partial g}{\partial y}}{-\frac{\partial g}{\partial z}}$

$$= \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right) - \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'} \right)\left(\frac{\partial g/\partial y}{\partial g/\partial z} \right) \right] \eta_1(x)\, dx$$

$\eta_1(x)$ is arbitrary so

$$\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\left(\frac{\partial g}{\partial y}\right)^{-1} = \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'}\right)\left(\frac{\partial g}{\partial z}\right)^{-1}$$

Each side of the equation is equal to a function of X : $-\lambda(x)$

$\uparrow$ Lagrange undetermined multiplier

Thus,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \lambda(x)\frac{\partial g}{\partial y} = 0$$

$$\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'} + \lambda(x)\frac{\partial g}{\partial z} = 0$$

Generalized:

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} + \sum_j \lambda_j(x)\frac{\partial g_j}{\partial y_j} = 0$$

Conservation of Energy

A closed system cannot depend explicitly on time

$$\frac{\partial L}{\partial t} = 0$$

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j$$

Lagrange's Equation:

$$\frac{\partial L}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}$$

Plugging Lagrange's Equation into $\frac{dL}{dt}$

$$\frac{dL}{dt} = \sum_j \dot{q}_j \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j$$

Using chain rule:

$$\frac{dL}{dt} - \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = 0$$

$$\frac{d}{dt}\left(L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = 0$$

Let

$$L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = -H$$

If potential energy is independent of $\dot{x}_{\alpha,i}$ or t then, $U = U(q_j)$ and $\frac{\partial U}{\partial \dot{q}_j} = 0$

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial (T-U)}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$$

Therefore,

$$(T-U) - \sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = -H$$

From an earlier (excluded proof) $\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T$

$$T-U - 2T = -H$$

$$\boxed{T+U = E = H = \text{constant}}$$

Conservation of Linear Momentum

For an infinitesimal translation of every radius vector $r_\alpha \to r_\alpha + \delta r$

Where $L = L(x_i, \dot{x}_i)$, $\delta r = \sum_i \delta x_i e_i$ changes L to

$$\delta L = \sum_i \frac{\partial L}{\partial x_i} \delta x_i + \sum_i \frac{\partial L}{\partial \dot{x}_i} \delta \dot{x}_i = 0$$

For an infinitesimal translation

$$\delta \dot{x}_i = \delta \frac{dx_i}{dt} = \frac{d}{dt} \delta x_i = 0$$

$$\delta L = \sum_i \frac{\partial L}{\partial x_i} \delta x_i = 0$$

Since δx_i is independent of displacement, δL only vanishes when $\frac{\partial L}{\partial x_i} = 0$

Lagrange's equation then becomes

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = 0 \quad \Rightarrow \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = 0$$

Therefore,

$\frac{\partial L}{\partial \dot{x}_i}$ is constant

$$\frac{\partial (T-U)}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i} \left(\frac{1}{2} m \sum_i \dot{x}_i^2\right) = m\dot{x}_i = p_i = \text{constant}$$

Conservation of Angular Momentum

Mechanical properties of a closed system are unaffected by orientation of the system

For an infinitesimal angle shift $\delta\theta$

$$\delta r = \delta\theta \times r$$

$$\delta \dot{r} = \delta\theta \times \dot{r}$$

$$\delta L = \sum_i \frac{\partial L}{\partial x_i}\delta x_i + \sum_i \frac{\partial L}{\partial \dot{x}_i}\delta \dot{x}_i = 0$$

As proven above:

$$p_i = \frac{\partial L}{\partial \dot{x}_i} \quad \text{so} \quad \dot{p}_i = \frac{\partial L}{\partial x_i}$$

$$\delta L = \sum_i \dot{p}_i \delta x_i + \sum_i p_i \delta \dot{x}_i = 0$$

$$\dot{p} \cdot \delta r + p \cdot \delta \dot{r} = 0$$

$$\dot{p}(\delta\theta \times r) + p(\delta\theta \times \dot{r}) = 0$$

$$\delta\theta \cdot (r \times p) + \delta\theta(\dot{r} \times p) = 0$$

$$\delta\theta[(r \times p) + (\dot{r} \times p)] = 0$$

Rewriting the expression

$$\delta\theta \cdot \frac{d}{dt}(r \times p) = 0$$

Since $\delta\theta$ is arbitrary we can simplify this expression

$$\boxed{\frac{d}{dt}(r \times p) = 0}$$

Kepler's Second Law

Consider a particle of mass m moving in a central force field

We know that angular momentum is conserved due to spherical symmetry

$L = r \times p$ = constant so r and p are normal to L

Lagrangian:

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$

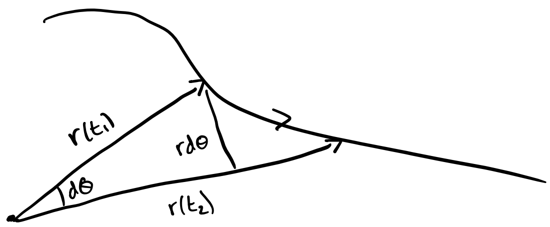
Since Lagrangian is cyclic in θ

$$\dot{p}_\theta = \frac{\partial L}{\partial \theta} = 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}$$

Therefore,

$$P_\theta = \frac{\partial L}{\partial \theta} = \mu r^2 \dot{\theta} = \text{constant}$$

Let $\ell = \mu r^2 \dot{\theta}$



Area traced out:

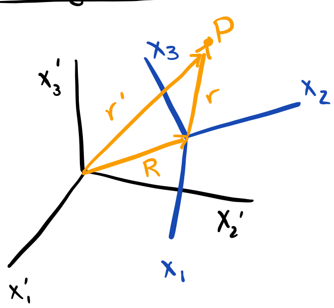
$$dA = \frac{1}{2} r^2 \, d\theta$$

Areal Velocity:

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

$$\boxed{= \frac{\ell}{2\mu} = \text{constant}}$$

Rotating Axis



x' → fixed axis

x → rotating axis

$$r' = R + r$$

For infinitesimal rotation of x_i system

$$(dr)_{\text{fixed}} = d\theta \times r$$

$$\left(\frac{dr}{dt}\right)_{\text{fixed}} = \frac{d\theta}{dt} \times r$$

$$\left(\frac{dr}{dt}\right)_{\text{fixed}} = \omega \times r$$

If P is rotating as well

$$\left(\frac{dr}{dt}\right)_{\text{fixed}} = \left(\frac{dr}{dt}\right)_{\text{rotating}} + \omega \times r$$

For r':

$$\left(\frac{dr'}{dt}\right)_{\text{fixed}} = \left(\frac{dR}{dt}\right)_{\text{fixed}} + \left(\frac{dr}{dt}\right)_{\text{fixed}}$$

$$= \left(\frac{dR}{dt}\right)_{\text{fixed}} + \left(\frac{dr}{dt}\right)_{\text{rotating}} + \omega \times r$$

Let $V_f = \left(\frac{dr'}{dt}\right)_{fixed}$: Velocity relative to fixed axis

$V = \left(\frac{dR}{dt}\right)_{fixed}$: Linear velocity of rotating axis relative to fixed axis

$V_r = \left(\frac{dr}{dt}\right)_{rotating}$: Velocity relative to rotating axis

ω : Angular velocity of rotating axis

$$V_f = V + V_r + \omega \times r$$

Centrifugal and Coriolis Forces

Newton's 2nd Law only applies in inertial reference frames

$$F = ma = m\left(\frac{dV_f}{dt}\right)_{fixed}$$

$$= m\left(\underbrace{\left(\frac{dV}{dt}\right)_{fixed}}_{\ddot{R}_f} + \underbrace{\left(\frac{dV_r}{dt}\right)_{fixed}}_{\left(\frac{dV_r}{dt}\right)_f = \left(\frac{dV_r}{dt}\right)_R + \omega \times V_r} + \dot{\omega} \times r + \underbrace{\omega \times \left(\frac{dr}{dt}\right)_{fixed}}_{= \omega \times \left(\frac{dr}{dt}\right)_{rotating} + \omega \times (\omega \times r)}\right)$$

$$= a_r + \omega \times V_r \qquad\qquad = \omega \times V_r + \omega \times (\omega \times r)$$

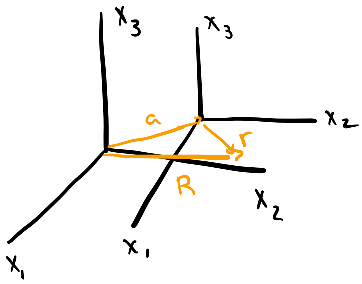
$$F = m\ddot{R}_f + ma_r + m\dot{\omega} \times r + 2m\,\omega \times V_r + m\omega \times (\omega \times r)$$

However, in the noninertial reference frame, the effective force

$$F_{eff} = ma_r$$

$$= F - \underbrace{m\ddot{R}_f}_{\text{translational acceleration}} - \underbrace{m\dot{\omega} \times r}_{\text{rotational acceleration}} - \underbrace{m\omega \times (\omega \times r)}_{\text{Centrifugal Force } (= m\omega^2 r \text{ when } \omega \perp r)} - \underbrace{2m\omega \times V_r}_{\text{Coriolis Force}}$$

Parallel Axis Theorem



$$R = a + r$$

$$X_i = a_i + x_i$$

$$J_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k X^2_{\alpha,k} - X_{\alpha,i} X_{\alpha j} \right)$$

$$= \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k (x_{\alpha,k} + a_k)^2 - (x_{\alpha i} + a_i)(x_{\alpha j} + a_j) \right)$$

$$= \underbrace{\sum_\alpha m_\alpha \left(\delta_{ij} \sum x^2_{\alpha k} - x_{\alpha i} x_{\alpha j} \right)}_{I_{ij}} + \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k (2x_{\alpha k} a_k + a_k^2) - (a_i x_{\alpha j} + a_j x_{\alpha i} + a_i a_j) \right)$$

$$= I_{ij} + \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k a_k^2 - a_i a_j \right) + \sum_\alpha m_\alpha \left(2\delta_{ij} \sum_k x_{\alpha,k} a_k - a_i x_{\alpha j} - a_j x_{\alpha i} \right)$$

$\sum_\alpha m_\alpha r_\alpha = 0$ since the center of mass is located at O

$$\sum_\alpha m_\alpha r_{\alpha k} = 0$$

$$J_{ij} = I_{ij} + \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k a_k^2 - a_i a_j \right)$$

$$I_{ij} = J_{ij} - M(a^2 \delta_{ij} - a_i a_j)$$

Lorentz Transformations

Assume the most simple coordinate transformation to account for Galilean shortcomings

$$x_1' = \gamma(x_1 - vt) \quad \text{and} \quad x_1 = \gamma'(x_1' + vt')$$

Laws of physics must be equivalent in each reference frame so $\gamma' = \gamma$

$$x_1' = \gamma(x_1 - vt) \quad \text{and} \quad x_1 = \gamma(x_1' + vt')$$

$$x_1 = \gamma(\gamma(x_1 - vt) + vt') = \gamma^2(x_1 - vt) + \gamma vt'$$

$$t' = \frac{x_1 - \gamma^2(x_1 - vt)}{\gamma v} = \frac{x_1}{\gamma v} - \frac{\gamma x_1}{v} + \gamma t = \frac{x_1}{\gamma v}\left(1 - \gamma^2\right) + \gamma t$$

Speed of light must be constant in each reference frame

$$x_1 = ct$$
$$x_1' = ct'$$

$$x_1 = \gamma(x_1' + vt') = \gamma\left(x_1' + \frac{v}{c}x_1'\right) \qquad x_1' = \gamma(x_1 - vt) = \gamma\left(x_1 - \frac{v}{c}x_1\right)$$

$$\frac{x_1}{x_1'} = \gamma\left(1 + \frac{v}{c}\right) \qquad \frac{x_1'}{x_1} = \gamma\left(1 - \frac{v}{c}\right)$$

$$\Rightarrow \frac{x_1'}{x_1} = \frac{1}{\gamma\left(1 + \frac{v}{c}\right)}$$

$$\frac{1}{\gamma\left(1 + \frac{v}{c}\right)} = \gamma\left(1 - \frac{v}{c}\right)$$

$$1 = \gamma^2\left(1 - \frac{v^2}{c^2}\right)$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Full Transformation:

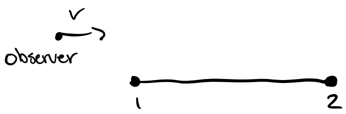
$$x_1' = \gamma(x_1 - vt) = \frac{x_1 - vt}{\sqrt{1 - v^2/c^2}}$$

$$x_2' = x_2$$

$$x_3' = x_3$$

$$t' = \frac{t - \frac{vx_1}{c^2}}{\sqrt{1 - v^2/c^2}}$$

Length Contraction



Length measured:

$$L' = X_1'(2) - X_1'(1) = \frac{X_1(2) - vt(2)}{\sqrt{1 - v^2/c^2}} - \frac{X_1(1) - vt(1)}{\sqrt{1 - v^2/c^2}} = \frac{(X_1(2) - X_1(1)) - v(t(2) - t(1))}{\sqrt{1 - v^2/c^2}}$$

$$t'(2) = t'(1) \qquad \frac{t(1) - \frac{vX_1(1)}{c^2}}{\sqrt{1 - v^2/c^2}} = \frac{t(2) - \frac{vX_1(2)}{c^2}}{\sqrt{1 - v^2/c^2}}$$

$$t(2) - t(1) = \frac{v}{c^2}\left(X_1(2) - X_1(1)\right)$$

$$L' = \frac{L - \frac{v^2}{c^2}L}{\sqrt{1 - v^2/c^2}} = L\left(\frac{1 - \frac{v^2}{c^2}}{\sqrt{1 - v^2/c^2}}\right) = L\sqrt{1 - v^2/c^2}$$

Time Dilation (fixed position)

$$\Delta t = t(2) - t(1)$$

Via lorentz transformations

$$\Delta t' = t'(2) - t'(1) = \frac{t(2) - \frac{vX_1(2)}{c^2} - \left[t(1) - \frac{vX_1(1)}{c^2}\right]}{\sqrt{1 - v^2/c^2}} = \frac{t(2) - t(1)}{\sqrt{1 - v^2/c^2}}$$

$$X_1(2) = X_1(1)$$

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - v^2/c^2}}$$

Relativistic Energy

$$W = \int_1^2 F \cdot dr = T_2 - T_1 \qquad F = \frac{dp}{dt} = \frac{d}{dt}(\gamma m u)$$

$$W = T = \int \frac{d}{dt}(\gamma m u) \cdot u\,dt = m \int_0^u u \; d(\gamma u)$$

Integration by parts

$$T = \gamma m u^2 - m\int_0^u \frac{u\,du}{\sqrt{1 - u^2/c^2}} = \gamma m c^2 - mc^2$$

Let $E_0 = mc^2$ represent rest energy

$$E = \gamma mc^2 = T + E_0$$

Sum of Kinetic and other energy

Relativistic Momentum and Energy

$$p = \gamma m u$$

$$p^2c^2 = \gamma^2 m^2 u^2 c^2 = \gamma^2 m^2 c^4 \left(\frac{u^2}{c^2}\right)$$

$$\left(\frac{u^2}{c^2}\right) = 1 - \frac{1}{\gamma^2}$$

$$p^2c^2 = \gamma^2 m^2 c^4 \left(1 - \frac{1}{\gamma^2}\right) = \gamma^2 m^2 c^4 - m^2c^4$$

$$= E^2 - E_0^2$$

$$\boxed{E^2 = p^2c^2 + E_0^2}$$

Spacetime Interval

Define Δs^2 s.t. it is invariant in all inertial systems in relative motion

$$\Delta s^2 = \sum_{j=1}^{3} (\Delta x_j)^2 - c^2\Delta t^2 \implies ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - c^2dt^2$$

$$\Delta s^2 = \Delta s'^2 = \sum_{j=1}^{3} (\Delta x_j')^2 - c^2\Delta t'^2$$