Complex Analysis Final Study Guide

Definitions + Facts

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

$$\arg(z) = \theta$$

$$|e^{i\theta}| = 1$$

$$\log z = \log|z| + i\arg(z)$$

$$\cosh(x) = \frac{e^{x}+e^{-x}}{2} \qquad \sinh(x) = \frac{e^{x}-e^{-x}}{2}$$

Cauchy Riemann Equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Harmonic Functions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ ← All analytic functions are harmonic via CR equations

Harmonic Conjugate:

Suppose u is harmonic, v is its harmonic conjugate if $h = u + iv$ is analytic and v is harmonic

Closed path:

starts and ends at the same point

Exact differentials:

For some holomorphic h, $dh = P\,dx + Q\,dy$

Closed differentials:

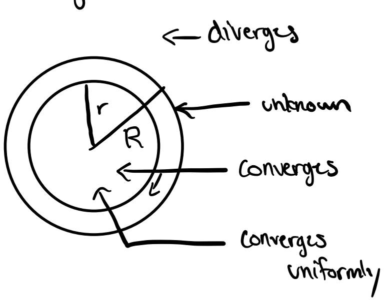
$$\frac{dP}{dy} = \frac{dQ}{dx}$$

Path independence, closed, and exact differentials

Path independent $\Leftrightarrow$ Exact $\Rightarrow$ closed

Star-shaped domains
path independent $\Leftrightarrow$ Exact $\Leftrightarrow$ closed

Radius of Convergence:



Power Series Expansion:

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k \quad , \quad |z-z_0| < \rho$$

$$a_k = \frac{f^{(k)}(z_0)}{k!}$$

when $R \geq \rho$

$$a_k = \frac{1}{2\pi i} \oint_{|s-z_0|=r} \frac{f(s)}{(s-z_0)^{k+1}}\, ds \qquad 0 \leq r \leq \rho$$

Isolated Singularities:

z_0 is an isolated singularity if $f(z)$ is analytic in some punctured disk around z_0

Removable Singularity:

z_0 is a removable singularity if $a_k = 0$ for all $k<0$

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k \qquad 0 < |z-z_0| < r$$

Pole Singularity:

An isolated singularity is a pole if there exists $N>0$ s.t. $a_{-N} \neq 0$ but $a_k = 0$ for all $k<-N$. N is the order of the pole

$$f(z) = \sum_{k=-N}^{\infty} a_k (z-z_0)^k$$

Meromorphic Functions:

$f(z)$ is analytic on D except for possible isolated singularities which are poles

Essential Singularity:

$a_k \neq 0$ for infinitely many $k<0$

Isolated Singularities at ∞:

f has isolated singularity if $g(w) = f(1/w)$ has an isolated singularity at $w=0$

$$f(z) = \sum_{k=-\infty}^{\infty} b_k z^k \qquad |z| > R$$

Removable: $b_k = 0$ for all $k>0$ $\rightarrow$ $f(z)$ is analytic at ∞

Essential: $b_k \neq 0$ for infinitely many $k>0$

Pole: Pole of order N at ∞ if $b_N \neq 0$ while $b_k = 0$ for $k>N$

Principle Part of $f(z)$

Negative powers; positive powers for ∞

Residue:

The residue of $f(z)$ at z_0 is a_{-1} or the coefficient of $\frac{1}{z-z_0}$

Residue Calculation Rules

Rule 1: $f(z)$ has a simple pole at z_0

$$\text{Res}[f(z), z_0] = \lim_{z \to z_0} (z-z_0) f(z)$$

Rule 2: $f(z)$ has a double pole at z_0

$$\text{Res}[f(z), z_0] = \lim_{z \to z_0} \frac{d}{dz}\left[(z-z_0)^2 f(z)\right]$$

Rule 3: $f(z)$ and $g(z)$ are analytic at z_0; $g(z)$ has a simple zero at z_0

$$\text{Res}\left[\frac{f(z)}{g(z)}, z_0\right] = \frac{f(z_0)}{g'(z_0)}$$

Rule 4: $g(z)$ is analytic and has a simple zero at z_0

$$\text{Res}\left[\frac{1}{g(z)}, z_0\right] = \frac{1}{g'(z_0)}$$

Principal Value:

$$PV \int_a^b f(x)\,dx = \lim_{\epsilon \to 0} \left(\int_a^{x_0-\epsilon} + \int_{x_0+\epsilon}^{b} f(x)\,dx \right)$$

Univalent: A function on domain D that is analytic and one-to-one

Winding Number:

$$W(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-z_0} = \frac{1}{2\pi} \int_\gamma d\arg(z-z_0)$$

$W(\gamma, \delta) = 0$ for all δ in the unbound component

winding number increases as δ crosses ∂D to D

Proofs / Theorems :

Green's Theorem

Let P and Q be continuously differentiable functions on $D \cup \partial D$.

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Path Independence and Exact differentials

$\int P\,dx + Q\,dy$ is independent of path in D iff $P\,dx + Q\,dy$ is exact, $dh = P\,dx + Q\,dy$

Proof: ← Proof incomplete

$\int P\,dx + Q\,dy$ is path independent $\Rightarrow$ $P\,dx + Q\,dy$ is exact

$$\int_a^b P\,dx + Q\,dy$$

Exact differentials are closed

Exact differentials : $dh = P\,dx + Q\,dy$

$$P = \frac{dh}{dx} \qquad Q = \frac{dh}{dy}$$

$$\frac{\partial P}{\partial y} = \frac{\partial h}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$ → closed ✓

Closed differentials are exact on star-shaped domains

Proof: define $h(B) = \int_A^B P\,dx + Q\,dy$

Let $B = (X_0, Y_0)$ and $C = (X_1, Y_0)$

$$\int_A^B + \int_B^C + \int_C^A (P\,dx + Q\,dy) = 0$$

$$\int_A^B - \int_A^C P\,dx + Q\,dy = \int_C^B P\,dx + Q\,dy$$

↲ Parameterizing path

$$h(B) - h(C) = \int_{X_1}^{X_0} P(t, Y_0)\,dt$$

From FTC:

Analagous proof

$$\frac{\partial h}{\partial x}(x_0,y_0) = P(x_0,y_0) \qquad \frac{\partial h}{\partial y}(x_0,y_0) = Q(x_0,y_0)$$

$Pdx + Q\,dy$ is exact ∎

Harmonic Conjugate on a star-shaped domain

$$V(B) = \int_A^B -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

Proof: ← Proof Incomplete

Mean Value Property

Let $h(z)$ be a continuous real valued function on a domain D. Let $z_0 \in D$ and suppose D contains the disk $\{|z-z_0|<\rho\}$

Average value of $h(z)$ on the circle $\{|z-z_0|=r\}$

$$A(r) = \int_0^{2\pi} h(z+re^{i\theta})\frac{d\theta}{2\pi} \qquad 0<r<\rho$$

As $r \to 0$ $A(r) \to h(z_0)$

If $u(z)$ is a harmonic function on a domain D and if the disk $\{|z-z_0|<\rho\}$ is contained in D, then

$$u(z_0) = \int_0^{2\pi} u(z_0+re^{i\theta})\frac{d\theta}{2\pi} \qquad 0<r<\rho$$

Average value at the boundary is its value at the center

$$0 = \oint_{|z-z_0|=r} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

closed differential $=0$ via green's thm

parameterizing on a circle

$x = r\cos\theta$ $\quad dx = -r\sin\theta$

$y = r\sin\theta$ $\quad dy = r\cos\theta$

$$0 = r\int_0^{2\pi}\left[\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right]d\theta = r\int_0^{2\pi}\frac{\partial u}{\partial r}(z_0+re^{i\theta})\,d\theta$$

Interchange differentiation and integration since u is smooth

$$0 = \frac{\partial}{\partial r} r\int_0^{2\pi} u(z_0+re^{i\theta})\,d\theta \xRightarrow{\text{divide by } 2\pi r} 0 = \frac{\partial}{\partial r}\int_0^{2\pi} u(z_0+re^{i\theta})\frac{d\theta}{2\pi}$$

For $0<r<\rho$

$$\frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{i\theta})\,d\theta \text{ is constant so as } r \to 0 \quad u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{i\theta})\,d\theta$$

Maximum Principle

Strict Maximum Principle:

Real Version: Let $u(z)$ be a real valued harmonic function on a domain D s.t. $u(z) \leq M$ $\forall z \in D$.

If $u(z_0) = M$ for some $z_0 \in D$ then $u(z) = M$ $\forall z \in D$.

Suppose $u(z_1) = M$

$$0 = \int_0^{2\pi} u(z_1) - u(z_1 + re^{i\theta}) \frac{d\theta}{2\pi} \qquad 0 < r < \rho$$

Since the integrand is non-negative and continuous this can only be true if the integrand is 0

$$u(z_1) = u(z_1 + re^{i\theta}) = M$$

Therefore there exists a disk centered around each point in the set $\{u(z) = M\}$. Therefore we can determine that it is an open set.

Since D is either $\{u(z) < M\}$ or $\{u(z) = M\}$ we know that one must be empty if the other is open. ∎

Complex Version:

Let h be a bounded complex-valued harmonic function on a domain D. If $|h(z)| \leq M$ for all $z \in D$, and $|h(z_0)| = M$ for some $z_0 \in D$ then $h(z_0)$ is constant on D.

Suppose $|h(z_0)| = M$. Let λ be a constant s.t. $\lambda h(z) = M$. λ is also unimodular

Let $u(z) = \text{Re}(\lambda h(z))$ where u is definitionally harmonic and real-valued.

$u(z_0) = M$ and by the real version of the strict maximum principle we find that

$u(z) = M$.

Since $|\lambda h(z)| \leq M$ and $u(z) = M$ we conclude that $\text{Im}(\lambda h(z)) = 0$.

Therefore $\lambda h(z)$ is constant and by extension $h(z)$ as well.

Maximum Principle:

Let $h(z)$ be a complex valued harmonic function on a bounded domain D such that $h(z)$ extends continuously to the boundary ∂D. If $|h(z)| \leq M$ $\forall z \in \partial D$, then $|h(z)| \leq M$ for all $z \in D$

Fundamental Theorem of Calculus for Analytic Functions

FTC Part 1: If $f(z)$ is continuous on domain D, and if $F(z)$ is primitive for $f(z)$

$$\int_A^B f(z)\,dz = F(B) - F(A)$$

$$F'(z) = \frac{\partial F}{\partial x} = \frac{1}{i}\frac{\partial F}{\partial y}$$

$$F(B)-F(A) = \int_A^B dF = \int_A^B \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = \int_A^B F'(z)\,(dx + i\,dy) = \int_A^B F'(z)\,dz$$

FTC Part 2: Let D be a star-shaped domain and let $f(z)$ be analytic on D. Then $f(z)$ has a primitive on D

and the primitive is unique up to adding a constant. The primitive of $f(z)$:

$$F(z) = \int_{z_0}^{z} f(\zeta)\,d\zeta \qquad z \in D \qquad \text{for a fixed point } z_0$$

Let $f(z) = u(z) + i v(z)$.

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{from C-R equations}$$

Therefore $u\,dx - v\,dy$ is closed and exact.

Since $u\,dx - v\,dy$ is exact we know $dU = u\,dx - v\,dy$ for a continuously differentiable function U.

U is also harmonic

$$\frac{\partial U}{\partial x^2} + \frac{\partial U}{\partial y^2} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \quad \text{via C-R equations}$$

For a harmonic function on a star-shaped domain we know there exists a harmonic conjugate V

$$G = U + iV$$

$$G' = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} \underset{\text{C-R equations}}{=} \frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = u + iv = f \qquad \blacksquare$$

Cauchy's Theorem

Lemma: A continuously differentiable function $f(z)$ on D is analytic iff the differential $f(z)dz$ is closed

$$f(z) = u + iv$$

$$f(z)dz = u+iv\,(dx+idy) = (u+iv)dx + (-v+iu)dy$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{from C-R equations}$$

$$\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} + -\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = 0 \quad \Rightarrow \quad f(z)dz \text{ is closed}$$

Let D be a bounded domain with piecewise smooth boundary. If $f(z)$ is analytic on D that extends smoothly to ∂D, then

$$\int_{\partial D} f(z)dz = 0$$

$$f(z) \text{ is analytic } \underset{\text{Lemma}}{\Rightarrow} f(z)dz \text{ is closed } \underset{\text{Green's Theorem}}{\Rightarrow} \int f(z)dz = 0$$

Cauchy Integral Formula

Let D be a bounded domain with a piecewise smooth boundary. If $f(z)$ is analytic on D and $f(z)$ extends smoothly to the boundary of D, then

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{z-w}\,dw \quad , \quad z \in D$$

For a fixed point z in D, let $D_\varepsilon = D \setminus \{|w-z| \le \varepsilon\}$ for small $\varepsilon > 0$

∂D_ε is then the union of ∂D and $\{|w-z| = \varepsilon\}$

Since $z \notin D_\varepsilon$, $\frac{f(w)}{w-z}$ is analytic on the domain

Applying Cauchy Formula:

$$\int_{\partial D_\varepsilon} \frac{f(w)}{w-z}\,dw = 0$$

orientation

$$\int_{\partial D_\varepsilon} \frac{f(w)}{w-z}\,dw = \int_{\partial D} \frac{f(w)}{w-z}\,dw - \int_{|w-z|=\varepsilon} \frac{f(w)}{w-z}\,dw \quad \Rightarrow \quad \int_{\partial D} \frac{f(w)}{w-z}\,dw = \int_{|w-z|=\varepsilon} \frac{f(w)}{w-z}\,dw$$

Change of Variable

$$w = z + \varepsilon e^{i\theta} \qquad dw = \varepsilon i e^{i\theta}\,d\theta$$

$$\int_0^{2\pi} \frac{f(z+\varepsilon e^{i\theta})}{\cancel{\varepsilon e^{i\theta}}} \cdot \cancel{\varepsilon e^{i\theta}}\, i\,d\theta$$

Mean value property:

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + \varepsilon e^{i\theta})\, d\theta$$

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w-z}\, dw \quad \blacksquare$$

Cauchy Integral Formula for Derivatives

Let D be a bounded domain with piecewise smooth boundary. If $f(z)$ is an analytic function on D that extends smoothly to the boundary of D, then $f(z)$ has complex derivatives of all orders on D.

$$f^{(m)}(z) = \frac{m!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{m+1}}\, dw \qquad z \in D,\ m \geq 0$$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{1}{\Delta z} \cdot \frac{1}{2\pi i} \left[\int_{\partial D} \frac{f(w)}{w - z - \Delta z}\, dw - \int_{\partial D} \frac{f(w)}{w-z}\, dw \right]$$

$$\frac{1}{2\pi i} \int_{\partial D} f(w) \cdot \frac{1}{(w-z-\Delta z)(w-z)}\, dw$$

As $\Delta z \to 0$

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^2}$$

Use induction and binomial expansion for higher order derivatives

Liouville's Theorem

Cauchy Estimate:

Suppose $f(z)$ is analytic for $|z - z_0| \leq \rho$. If $|f(z)| \leq M$ for $|z - z_0| = \rho$

then

$$|f^{(m)}(z_0)| \leq \frac{m!}{\rho^m} M$$

$$f^{(m)}(z_0) = \frac{m!}{2\pi i} \int_{|z-z_0|=\rho} \frac{f(z)}{(z-z_0)^{m+1}}\, dz$$

Parameterize z

$$z = z_0 + \rho e^{i\theta} \qquad dz = \rho \cdot e^{i\theta} i\, d\theta$$

$$f^{(m)}(z_0) = \frac{m!}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{\rho^{m+1} e^{(m+1)i\theta}} \rho e^{i\theta} i\, d\theta$$

$$= \frac{m!}{2\pi} \cdot \int_0^{2\pi} \frac{f(z_0+\rho e^{i\theta})}{\rho^m e^{mi\theta}} = \frac{m!}{\rho^m} \cdot \overbrace{\int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{e^{mi\theta}} \frac{d\theta}{2\pi}}^{\leq M}$$

Let $f(z)$ be an analytic function on the complex plane. If $f(z)$ is bounded, then it is constant

Suppose $|f(z)| \leq M \quad \forall z \in \mathbb{C}$

From the cauchy estimate:

$$|f'(z_0)| \leq \frac{M}{\rho} \quad \text{for arbitrary disk size } \rho \text{ and } z_0$$

As $\rho \to \infty$, $|f'(z)| \leq 0$ so $f'(z) = 0$ and f is constant ∎

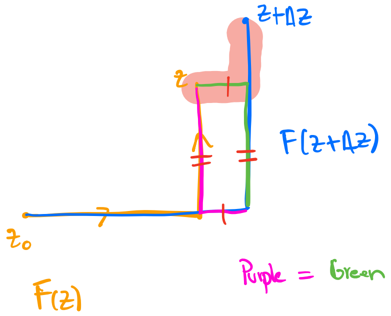
Morera's Theorem

Let $f(z)$ be a continuous function on a domain D. If $\int_{\partial R} f(z)dz = 0$ for every closed rectangle R contained in D with sides parallel to the coordinate axis, then $f(z)$ is analytic on D.

Assume D is a disk with center at z_0

Let $F(z) = \int_{z_0}^{z} f(\zeta)d\zeta \quad z \in D$ where the integration path is horizontal and then vertical

$$F(z+\Delta z) - F(z) = \underbrace{\int_{z}^{z+\Delta z} f(\zeta)d\zeta}_{\text{red portion}}$$



Purple = Green by assumption

Add constant $f(z)$ and subtract $f(z)$ to evaluate RHS

$$F(z+\Delta z) - F(z) = f(z)\int_{z}^{z+\Delta z} d\zeta + \int_{z}^{z+\Delta z} f(\zeta) - f(z)\, d\zeta$$

$$= f(z)\,\Delta z + \int_{z}^{z+\Delta z} f(\zeta) - f(z)\, d\zeta$$

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(\zeta) - f(z)\, d\zeta$$

Applying ML-estimates

$$\int_{z}^{z+\Delta z} f(\zeta) - f(z)\, d\zeta \leq 2|\Delta z| \cdot M_\varepsilon \quad \text{where } M_\varepsilon \text{ is maximum of } |f(\zeta) - f(z)|$$

Therefore, $\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| \leq 2M_\varepsilon$ which approaches 0 as $\Delta z \to 0$.

$f(z)$ is therefore analytic since $f(z)$ is continuous, $F(z)$ is analytic and $F'(z) = f(z)$.

Goursat's Theorem

If $f(z)$ is a complex valued function on a domain D such that $f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists at each point of D

then $f(z)$ is analytic on D.

"Goursat's Theorem is as useless as it is aesthetically pleasing"

Let R be a closed rectangle in D. Divide R into 4 sub rectangles

$$\int_{\partial R} f(z)\,dz = \int_{\partial R_1} + \int_{\partial R_2} + \int_{\partial R_3} + \int_{\partial R_4} f(z)\,dz$$

So we can say for at least one sub rectangle R_1

$$\left|\int_{\partial R_1} f(z)dz\right| \geq \frac{1}{4}\left|\int_{\partial R} f(z)dz\right|$$

Repeating this procedure n times we find

$$\left|\int_{\partial R_n} f(z)dz\right| \geq \frac{1}{4}\left|\int_{\partial R_{n-1}} f(z)dz\right| \geq \cdots \geq \frac{1}{4^n}\left|\int_{\partial R} f(z)\,dz\right|$$

Eventually R_n approaches to some point in z_0 which we know to be differentiable

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| \leq \varepsilon_n \qquad z \in R_n$$

$$\varepsilon_n \to 0 \quad \text{as} \quad n \to \infty$$

If L is the length of ∂R then $\partial R_n = \frac{L}{2^n}$

$$|f(z)-f(z_0) - f'(z_0)(z-z_0)| \leq \varepsilon_n |z-z_0| \leq \frac{2\varepsilon_n L}{2^n}$$

ML-estimate

$$\left|\int_{\partial R_n} f(z)dz\right| \overset{?}{=} \left|\int_{\partial R_n} [f(z)-f(z_0)-f'(z_0)(z-z_0)]dz\right| \leq \frac{2\varepsilon_n L}{2^n} \cdot \left(\frac{L}{2^n}\right) = \frac{2L^2}{4^n}\varepsilon_n$$

$$\left|\int_{\partial R} f(z)dz\right| \leq 4^n\left|\int_{\partial R_n} f(z)dz\right| \leq 2L^2\varepsilon_n$$ but as $\varepsilon_n \to 0$ with $n \to \infty$ we find $\int_{\partial R} f(z)dz = 0$

Now apply Morera's theorem to finalize $f(z)$ is analytic

Weierstrass M-Test

Suppose $M_k \geq 0$ and $\sum M_k$ converges. If $g_k(x)$ are complex valued functions on a set E such that $|g_k(x)| \leq M_k$ for all $x \in E$, then $\sum g_k(x)$ converges uniformly on E.

For each fixed x we have that $\sum g_k(x)$ is absolutely convergent and $\sum |g_k(x)| \leq \sum M_k$

$\sum g_k(x)$ converges to some $g(x)$ s.t. $|g(x)| \leq \sum |g_k(x)| \leq \sum M_k$

Considering the tail of the series

$$|g(x) - S_n(x)| = \sum_{k=n+1}^{\infty} g_k(x) \leq \sum_{k=n+1}^{\infty} M_k$$

If $\varepsilon_n = \sum_{k=n+1}^{\infty} M_k$ as $n \to \infty$ $\varepsilon_n \to 0$ and we find $S_n(x)$ converges uniformly to $g(x)$

Ratio Test

If $\left|\frac{a_k}{a_{k+1}}\right|$ has a limit as $k \to \infty$, either finite or $+\infty$, then the limit is the radius of convergence R of $\sum a_k z^k$

$$R = \lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right|$$

Let $L = \lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right|$. If $r < L$ then $\left|\frac{a_k}{a_{k+1}}\right| > r$ for $k \geq N$.

Then $|a_k| > r|a_{k+1}|$ for $k \geq N$

$$|a_N| r^N \geq |a_{N+1}| r^{N+1} \geq |a_{N+2}| r^{N+2} \geq \dots \qquad |a_k| r^k \text{ is bounded}$$

By definition of R we have that for $r \geq R$ and since $r < L$ is arbitrary, $L \leq R$

Suppose $s > L$. Then $\left|\frac{a_k}{a_{k+1}}\right| < s$ and eventually for $k \geq N$, $|a_k| < s|a_{k+1}|$

$$|a_N| s^N \leq |a_{N+1}| s^{N+1} \leq |a_{N+2}| s^{N+2} \leq \dots$$ and $a_k z^k$ does not converge to 0 for $|z| \geq s$

Since $s \geq R$ and $s > L$ is arbitrary we conclude $L \geq R$ and finally $L = R$

Cauchy-Hadamard Formula

If $\sqrt[k]{|a_k|}$ has a limit as $k \to \infty$, then the radius of convergence of $\sum a_k z^k$ is given by

$$R = \frac{1}{\limsup \sqrt[k]{|a_k|}}$$

lim sup is the value at which there are finitely many terms greater than t and infinitely $< t$

Proof of simplified version: $R = \frac{1}{\lim \sqrt[k]{|a_k|}}$

If $r > \frac{1}{\lim \sqrt[k]{|a_k|}}$, then $\sqrt[k]{|a_k|}\, r > 1$ eventually.

$|a_k| r^k > 1$ and terms of $\sum a_k z^k$ do not converge to 0 for $|z| = r$ and $r \geq R$

If $r < \frac{1}{\lim \sqrt[k]{|a_k|}}$ we see that $|a_k| r^k < 1$ is bounded and we see that $R = \frac{1}{\lim \sqrt[k]{|a_k|}}$

Identity Principle

If D is a domain and $f(z)$ is an analytic function on D that is not identically zero, then the zeros of $f(z)$ are isolated

Let U be the set of all $z \in D$ such that $f^{(m)}(z) = 0$ for all $m \geq 0$.

If $z_0 \in U$, then the power series expansion $f(z) = \sum a_k (z - z_0)^k$ has $a_k = \frac{f^{(k)}(z_0)}{k!} = 0$

Therefore, $f(z) = 0$ for z belonging to a disk centered at z_0

↳ points in the disk also exist in U

↳ U is an open set

If $z_0 \in D/U$ then $f^{(k)}(z_0) \neq 0$ and $f^{(k)}(z) \neq 0$ for some disk around z_0 showing that $D \setminus U$ is also open.

Since D is connected, either $U = D$ or U is empty.

If $U = D$, we find that $f(z) = 0$ on D which contradicts our hypothesis so U must be empty.

Suppose z_0 is a zero of $f(z)$ with finite order N.

$f(z) = (z - z_0)^N h(z)$ where $h(z)$ is analytic at z_0 and $h(z_0) \neq 0$

For small $\rho > 0$ we have $h(z) \neq 0$ for $|z - z_0| < \rho$, thus each zero of $f(z)$ is isolated.

Uniqueness Principle

If $f(z)$ and $g(z)$ are analytic on a domain D, and if $f(z) = g(z)$ for z belonging to a set that has a nonisolated point then

$f(z) = g(z)$ for all $z \in D$.

$f(z) - g(z)$ has non isolated zero so $f(z) - g(z)$ is identically 0.

Final Exam

Laurent Decomposition

Suppose $0 \leq \rho < \sigma \leq +\infty$ and suppose $f(z)$ is analytic for $\rho < |z - z_0| < \sigma$. Then $f(z)$ can be decomposed as a sum

$f(z) = f_0(z) + f_1(z)$

where $f_0(z)$ is analytic for $|z - z_0| < \sigma$ and $f_1(z)$ is analytic for $|z - z_0| > \rho$ and at ∞. If $f_1(\infty) = 0$ then the decomposition is unique

Proof:

Suppose $f(z)$ is analytic for $|z - z_0| < \sigma$, then $f(z) = f_0(z)$ and $f_1(z) = 0$

Similarly if $f(z)$ is analytic for $|z - z_0| > \rho$ and vanishes at ∞, then $f(z) = f_1(z)$ with $f_0(z) = 0$

Uniqueness Argument:

Suppose $f(z) = g_0(z) + g_1(z)$ is another decomposition

$f(z) - f(z) = f_0(z) + f_1(z) - g_0(z) - g_1(z) = 0$

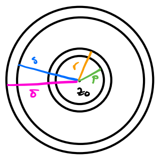
$g_0(z) - f_0(z) = f_1(z) - g_1(z) \qquad \rho < |z - z_0| < \sigma$

If $h(z) = g_0(z) - f_0(z)$ for $|z - z_0| < \sigma$ and $h(z) = f_1(z) - g_1(z)$ for $|z - z_0| > \rho$ we define $h(z)$ as an entire function where as $z \to \infty$ $h(z) \to 0$.

By Liouville's theorem we find $h(z) = 0$ and as a consequence, uniqueness of decomposition

Finding the decomposition

Choose r and s such that $\rho < r < s < \sigma$



Applying Cauchy Integral Formula

$$f(z) = \frac{1}{2\pi i} \oint_{|w-z_0|=s} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i} \oint_{|w-z_0|=r} \frac{f(w)}{w-z}\,dw \qquad r < |z-z_0| < s$$

Splitting $f(z)$ by components on the annulus

$$f_0(z) = \frac{1}{2\pi i} \oint_{|w-z_0|=s} \frac{f(w)}{w-z}\,dw \qquad |w-z_0| < s$$

$$f_1(z) = -\frac{1}{2\pi i} \oint_{|w-z_0|=r} \frac{f(w)}{w-z}\,dw \qquad |w-z_0| > r$$

By uniqueness argument we can assert $f_0(z)$ and $f_1(z)$ defined for r and s apply to ρ and σ

Laurent Series Expansion

Suppose $0 \le \rho < \sigma \le \infty$ and suppose $f(z)$ is analytic for $\rho < |z-z_0| < \sigma$. Then $f(z)$ has a Laurent series expansion that converges absolutely at each point of the annulus and that converges uniformly on each subannulus $r \le |z-z_0| \le s$ where $\rho < r < s < \sigma$. The coefficients are uniquely determined by $f(z)$ for any fixed r, $\rho < r < \sigma$

Expansion : $$f(z) = \sum_{-\infty}^{\infty} a_k (z-z_0)^k \qquad \rho < |z-z_0| < \sigma$$

Coefficients : $$a_n = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad -\infty < n < \infty$$

Proofs :

Suppose $f(z) = f_0(z) + f_1(z)$ for $\rho < |z-z_0| < \sigma$

$$f_0(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k \qquad |z-z_0| < \sigma$$

Series converges uniformly for $|z-z_0| \le s$

$$f_1(z) = \sum_{k=-\infty}^{-1} a_k (z-z_0)^k \qquad |z-z_0| > \rho$$

Satisfies $f_1(z) \to 0$ as $z \to$

Converges uniformly for $|z-z_0| \ge r$

Therefore,

$$f(z) = f_0(z) + f_1(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k \qquad \rho < |z-z_0| < \sigma$$ ← Laurent Expansion

Consider $\frac{f(z)}{(z-z_0)^{n+1}}$ integrated over $|z-z_0| = r$

$$\oint_{|z-z_0|=r} \frac{1}{(z-z_0)^{n+1}} f(z)\,dz = \oint \frac{1}{(z-z_0)^{n+1}} \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k\,dz = \sum_{k=-\infty}^{\infty} a_k \oint_{|z-z_0|=r} (z-z_0)^{k-n-1}\,dz$$

Swap since Series converges uniformly

Integral of $(z-z_0)^m = 2\pi i$ when $m=-1$ and 0 otherwise

Series then becomes $2\pi i a_n$

$$a_n = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad -\infty < n < \infty$$

Tail of Laurent decomposition with positive powers converges on the largest open disk centered at z_0 to which $f(z)$ extends to be analytic

Tail of Laurent decomposition with negative powers converges on the largest exterior domain to which $f_1(z)$ extends analytically

Riemann's Theorem on Removable Singularities

Let z_0 be an isolated singularity of $f(z)$. If $f(z)$ is bounded near z_0, then $f(z)$ has a removable singularity at z_0.

Suppose $|f(z)| \leq M$ for z near z_0 and small $r > 0$

Applying ML estimates

$$a_n = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} \, dz$$

$$|a_n| \leq \frac{1}{2\pi i} \cdot 2\pi r \cdot \frac{M}{r^{n+1}} = \frac{M}{r^n}$$

If $n < 0$, as $r \to 0$ $\frac{M}{r^n}$ approaches 0. Therefore $a_n = 0$ for $n < 0$ and therefore it is removable.

Poles and Limits

Let z_0 be an isolated singularity of $f(z)$. Then z_0 is a pole if and only if $|f(z)| \to \infty$ as $z \to z_0$

$\Rightarrow$ Suppose f has a pole of order N at z_0

$g(z) = (z-z_0)^N f(z)$ is analytic at non zero at z_0

$$|f(z)| = |z-z_0|^{-N} |g(z)| \to \infty \quad \text{as} \quad z \to z_0$$

$\Leftarrow$ Suppose $|f(z)| \to \infty$ as $z \to z_0$

Then $f(z) \neq 0$ for z near z_0, therefore $h(z) = \frac{1}{f(z)}$ is analytic on a punctured disk around z_0.

$h(z) \to 0$ as $z \to z_0$ and by Riemann's Theorem $h(z)$ is analytic at z_0 where $h(z_0) = 0$

N is the order of the zero of $h(z)$ at z_0, so $f(z) = \frac{1}{h(z)}$ has a pole of order N at z_0.

Casorati-Weierstrauss Theorem

Suppose z_0 is an essential isolated singularity of $f(z)$. Then for every complex number w_0, there is a sequence $z_n \to z_0$ such that $f(z_n) \to w_0$

Proof of contrapositive: Suppose there is some complex number w_0 that is not the limit of $f(z)$ as $z \to z_0$.

Then there exists some $\varepsilon > 0$ s.t. $|f(z) - w_0| > \varepsilon$ for all z near z_0. Therefore, $h(z) = \frac{1}{f(z) - w_0}$ is bounded for z near z_0. Applying Riemann's theorem we find that $f(z)$ has a removable singularity at z_0.

$h(z) = (z-z_0)^N g(z)$ for $N \geq 0$ and some analytical function $g(z)$ where $g(z_0) \neq 0$

$$\frac{1}{f(z) - w_0} = (z-z_0)^N g(z)$$

$$f(z) = (z-z_0)^{-N} \cdot \frac{1}{g(z)} + w_0$$

If $N = 0$ $f(z)$ is analytic at z_0, otherwise if $N > 0$ $f(z)$ has a pole of order N at z_0.

Meromorphic Functions on $\mathbb{C}^*$

A meromorphic function on the extended complex plane is rational

Meromorphic function must have a finite number of poles, otherwise they would not be isolated

Define $P_\infty(z)$ to be $f(\infty)$ if f is analytic at ∞, otherwise $P_\infty(z)$ is the priciple part of $f(z)$ at ∞

$$f(z) - P_\infty(z) \to 0 \quad \text{as} \quad z \to \infty$$

For poles $z_1 \ldots z_m$

$$P_k(z) = \frac{\alpha_1}{z-z_k} + \frac{\alpha_2}{(z-z_k)^2} + \ldots \frac{a_n}{(z-z_k)^n} \quad \leftarrow \text{Principle part of } f(z) \text{ at } z_k$$

Consider

$$g(z) = f(z) - P_\infty(z) - \sum_{j=1}^{m} P_j(z)$$

$g(z)$ is an entire function since all trouble areas (singularities) are accounted for

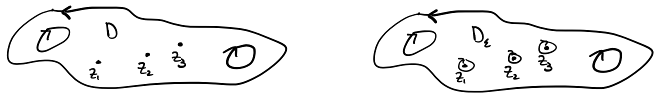
$g(z) \to 0$ as $z \to \infty$ so by Liouville's Theorem we get that $g(z) = 0$

$$f(z) = P_\infty(z) + \sum_{j=1}^{m} P_j(z) \quad \leftarrow \text{Partial fraction decomposition}$$

Residue Theorem

Let D be a bounded domain in the complex plane with piecewise smooth boundary. Suppose that $f(z)$ is analytic on $D \cup \partial D$ except for a finite number of isolated singularities $z_1 \dots z_m$ in D. Then

$$\int_{\partial D} f(z)\,dz = 2\pi i \sum_{j=1}^{m} \mathrm{Res}\left[f(z), z_j\right]$$



Let D_ε be the domain obtained by punching out small disks around each singularity

By definition we find that for the integral around each disk U_j

$$\int_{\partial U_j} f(z)\,dz = 2\pi i \,\mathrm{Res}\left[f(z), z_j\right]$$

From Cauchy's Theorem we have

$$0 = \int_{\partial D_\varepsilon} f(z)\,dz = \int_{\partial D} f(z)\,dz - \sum_{j=1}^{m} \int_{\partial U_j} f(z)\,dz = \int_{\partial D} f(z)\,dz - 2\pi i \,\mathrm{Res}\left[f(z), z_j\right]$$

$$\int_{\partial D} f(z)\,dz = \sum_{j=1}^{m} 2\pi i \,\mathrm{Res}\left[f(z), z_j\right]$$

Fractional Residue Theorem

If z_0 is a simple pole of $f(z)$, and C_ε is an arc of the circle $\{|z-z_0| = \varepsilon\}$ of angle α, then

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon} f(z)\,dz = \alpha i \,\mathrm{Res}\left[f(z), z_0\right]$$

Write $f(z) = \frac{A}{z-z_0} + g(z)$ where A is $\mathrm{Res}\left[f(z), z_0\right]$ and $g(z)$ is analytic at z_0

$$\int_{C_\varepsilon} \frac{A}{z-z_0} + g(z)\,dz = \int_{\theta_0}^{\theta_0+\alpha} \left(\frac{A}{\varepsilon e^{i\theta}} + g(z_0+\varepsilon e^{i\theta})\right) \cdot \varepsilon i e^{i\theta}\,d\theta = iA \int_{\theta_0}^{\theta_0+\alpha} d\theta + \underbrace{\int_{\theta_0}^{\theta_0+\alpha} g(z_0+\varepsilon e^{i\theta}) \cdot \varepsilon i e^{i\theta}\,d\theta} = iA \int_{\theta}^{\theta+\alpha} d\theta = \alpha i A = \alpha i \,\mathrm{Res}\left[f(z), z_0\right]$$

parameterize $z = z_0 + \varepsilon e^{i\theta}$

$dz = \varepsilon i e^{i\theta}\,d\theta$

ML-estimate: g is bounded near z_0

$\leq 2\pi\varepsilon \cdot M$

$\to 0$ as $\varepsilon \to 0$

Jordan's Lemma

If T_R is a semicircular contour $z(\theta) = Re^{i\theta}$ $0 \leq \theta \leq \pi$, in the upper half-plane, then

$$\int_{T_R} |e^{iz}|\,|dz| < \pi$$

$$z(\theta) = Re^{i\theta} \qquad \left|e^{iz}\right| = \left|e^{iR(\cos\theta + i\sin\theta)}\right| = e^{-R\sin\theta}$$

$$|dz| = R$$

$$\int_{T_R} |e^{iz}|\,|dz| = \int_0^\pi e^{-R\sin\theta} \cdot R\,d\theta < \pi \implies \int_0^\pi e^{-R\sin\theta}\,d\theta < \frac{\pi}{R}$$

Proving Estimate:

$$\sin\theta \geq \frac{2\theta}{\pi} \qquad 0 \leq \theta \leq \pi/2$$

$$\int_0^\pi e^{-R\sin\theta}\,d\theta = 2\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta \leq 2\int_0^{\pi/2} e^{-2R\theta/\pi}\,d\theta = \frac{\pi}{R}\int_0^R e^{-t}\,dt < \frac{\pi}{R}\int_0^\infty e^{-t}\,dt = \frac{\pi}{R}$$

$$t = \frac{2R\theta}{\pi}$$

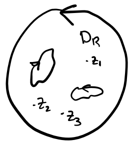
$$dt = \frac{2R}{\pi}\,dt$$

Residue Theorem for Exterior Domains

Let D be an exterior domain with a piecewise smooth boundary. Suppose $f(z)$ is analytic on $D \cup \partial D$ except for a finite number of isolated singularities $z_1 \dots z_m$ in D.

Let a_{-1} be the coefficient of $\frac{1}{z}$ in the laurent expansion $f(z) = \sum a_k z^k$ that converges for $|z| > R$. Then

$$\int_{\partial D} f(z)dz = -2\pi i a_{-1} + 2\pi i \sum_{j=1}^{m} \text{Res}[f(z), z_j]$$



Apply residue theorem to the bounded domain D_R where $z \in D$ s.t. $|z| < R$

$$\int_{\partial D} f(z)dz + \underbrace{\overset{?}{\int_{|z|=R}} f(z)dz}_{\text{evaluates to } 2\pi i a_{-1} \text{ when integrated term by term}} = \int_{\partial D_R} f(z)dz = 2\pi i \sum_{j=1}^{m} \text{Res}[f(z), z_j]$$

$$\int_{\partial D} f(z)dz = -2\pi i a_1 + 2\pi i \sum_{j=1}^{m} \text{Res}[f(z), z_j]$$

Argument Principle

Let D be a bounded domain with piecewise smooth boundary ∂D and let $f(z)$ be a meromorphic function on D that extends to be analytic on ∂D, such that $f(z) \neq 0$ on ∂D. Then

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)} dz = N_0 - N_\infty$$

where N_0 is the number of zeros of $f(z)$ in D and N_∞ is the number of poles of $f(z)$ in D, counting multiplicities

For a pole of $f(z)$ at z_0

$$f(z) = (z-z_0)^N g(z)$$

$$\frac{f'(z)}{f(z)} = \frac{N(z-z_0)^{N-1} g(z) + (z-z_0)^N g'(z)}{(z-z_0)^N g(z)} = \frac{N}{z-z_0} + \underbrace{\frac{g'(z)}{g(z)}}_{\text{Analytic function}}$$

$\frac{f'(z)}{f(z)}$ has a simple pole at z_0 with residue N

Residue theorem on $\frac{f'(z)}{f(z)}$ is the sum of residues which are $+N$ for each zero and $-N$ for each pole.

Thus, $\frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)} dz = N_0 - N_\infty$

Consider the logarithmic integral

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_\gamma d \log f(z) = \frac{1}{2\pi i} \int_\gamma d \log |f(z)| + \frac{1}{2\pi i} \int_\gamma d \arg(f(z))$$

exact

parameterizing $\gamma(t) = x(t) + iy(t)$ for $a \leq t \leq b$

$$\int_\gamma d \log |f(z)| = \log |f(\gamma(b))| - \log |f(\gamma(a))| \longleftarrow 0 \text{ for closed curve}$$

$$\int_\gamma d \arg(f(z)) = \arg f(\gamma(b)) - \arg f(\gamma(a))$$

Rouché's Theorem

Let D be a bounded domain with piecewise smooth boundary ∂D. Let $f(z)$ and $h(z)$ be analytic on $D \cup \partial D$. If $|h(z)| < |f(z)|$ for $z \in \partial D$, then $f(z)$ and $f(z)+h(z)$ have the same number of zeros in D, counting multiplicities.

$f(z)$ and $f(z)+h(z) \neq 0$ on ∂D

$$f(z)+h(z) = f(z)\left[1+\frac{h(z)}{f(z)}\right]$$

$$\arg(f(z)+h(z)) = \arg(f(z)) + \arg\left(1+\frac{h(z)}{f(z)}\right)$$

Since $|h(z)| < |f(z)|$ $\quad 1+\frac{h(z)}{f(z)}$ must exist in the right half-plane so $\arg\left(1+\frac{h(z)}{f(z)}\right) = 0$

Therefore $\arg(f(z)+h(z)) = \arg(f(z))$

Thus by the argument principle they have the same number of zeros

★ Hurwitz's Theorem

Suppose $\{f_k(z)\}$ is a sequence of analytic functions on a domain D that converges normally on D to $f(z)$ and suppose that $f(z)$ has zero of order N at z_0. Then there exists $\rho > 0$ such that for k large, $f_k(z)$ has exactly N zeros in the disk $\{|z-z_0|<\rho\}$ counting multiplicity, and these zeros converge to z_0 as $k \to \infty$

Let $\rho > 0$ be sufficiently small s.t. $\{|z-z_0| \le \rho\}$ is contained in D and so $f(z) \neq 0$ for $0 < |z-z_0| \le \rho$.

Choose $\delta > 0$ s.t. $|f(z)| \ge \delta$ on the circle $|z-z_0| = \rho$. Since $f_k(z)$ converges uniformly to $f(z)$ for $|z-z_0| \le \rho$, for k large we have $|f_k(z)| > \delta/2$ for $|z-z_0|=\rho$ and $\frac{f_k'(z)}{f_k(z)}$ converges uniformly to $\frac{f'(z)}{f(z)}$ for $|z-z_0|=\rho$

$$\underbrace{\frac{1}{2\pi i}\int_{|z-z_0|=\rho} \frac{f_k'(z)}{f_k(z)}\,dz}_{N_k} \longrightarrow \underbrace{\frac{1}{2\pi i}\int_{|z-z_0|=\rho} \frac{f'(z)}{f(z)}\,dz}_{N} \qquad \Rightarrow \quad N_k \to N$$

★ Open Mapping for Analytic Functions

If $f(z)$ is analytic on a domain D and $f(z)$ is not constant, then $f(z)$ maps open sets to open sets, that is, $f(U)$ is open for each open subset U of D

Let $w_0 \in f(U)$ say $w_0 = f(z_0)$.

Consider $f(z_0) - w_0$ which has a zero of order m at z_0. Since the zeros are isolated we can construct a disk of radius ρ so that $f(z) - w_0 \neq 0$ for $0 < |z-z_0| \le \rho$

★ Inverse Function Theorem

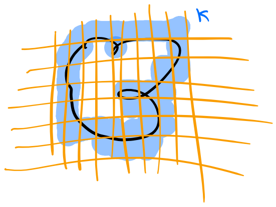
Suppose $f(z)$ is analytic for $|z-z_0| \le \rho$ and satisfies $f(z_0) = w_0$, $f'(z_0) = 0$ and $f(z) \neq w_0$ for $0 < |z-z_0| \le \rho$. Let $\delta > 0$ be chosen s.t. $|f(z) - w_0| \ge \delta$ for $|z-z_0| = \rho$. Then for w s.t. $|w-w_0| < \delta$ there is a unique z satisfying $|z-z_0| < \rho$ and $f(z) = w$. $z = f^{-1}(w)$

$$f^{-1}(w) = \frac{1}{2\pi i}\int_{|\zeta-z_0|=\rho} \frac{\zeta f'(\zeta)}{f(\zeta)-w}\,d\zeta \qquad |w-w_0|<\delta$$

Generalized Cauchy Formula

If $f(z)$ is analytic on a domain D, then $\int_\gamma f(z)dz = 0$ for each closed path γ in D s.t. $W(\gamma, s) = 0$ for all $s \in \mathbb{C} \setminus D$

Choose $\delta > 0$ small s.t. every point of Γ has distance at least 4δ from any point of $\mathbb{C}\setminus D$. Then divide D into a grid of squares of side length δ



Let K be the union of all squares in the grid with a point less than δ away from Γ. $U = K \setminus \partial K$ so $\partial U = \partial K$

Using cauchy integral formula

$$z \in \Gamma \quad f(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(s)}{s - z} ds$$

$$\int_\gamma f(z)dz = \frac{1}{2\pi i} \int_{\partial U} \left[\int_\gamma \frac{1}{s-z} dz \right] f(s) ds = -\int_{\partial U} W(\gamma, s) f(s) ds$$

Winding number is 0 so $\int_\gamma f(z) dz = 0$

Generalized Cauchy Integral Formula

Let $f(z)$ be analytic on a domain D and let γ be a closed path in D with trace $\Gamma = \gamma([a,b])$. If $W(\gamma, s) = 0$ for all $s \in \mathbb{C} \setminus D$, then

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz = W(\gamma, z_0) f(z_0) \qquad z_0 \in D \setminus \Gamma$$

Let $g(z) = \dfrac{f(z) - f(z_0)}{z - z_0}$. $\int_\gamma g(z) dz = 0$

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \int \frac{f(z_0)}{z - z_0} dz = W(\gamma, z_0) f(z_0)$$

Schwarz Lemma

Let $f(z)$ be analytic for $|z| < 1$. Suppose $|f(z)| \le 1$ for all $|z| < 1$ and $f(0) = 0$. Then

$$|f(z)| \le |z|, \qquad |z| < 1$$

If the equality holds for some z, $z_0 \ne 0$ then $f(z) = \lambda z$ for some constant λ of unit modulus

Write $f(z) = z g(z)$ where $g(z)$ is analytic.

Let $r < 1$. If $|z| = r$ then $|g(z)| = \dfrac{|f(z)|}{|z|} \le \dfrac{1}{r}$. By the maximum principle $|g(z)| \le \frac{1}{r}$ for all z satisfying $|z| \le r$

As $r \to 1$, $|g(z)| \le 1$ for all $|z| < 1$.

If $|f(z_0)| = |z_0|$ for some $z_0 = 0$ then $|g(z_0)| = 1$ and by strict maximum principle $g(z)$ is constant.

$g(z) = \lambda$ and $f(z) = \lambda z$

Theorem can easily be extended to other disks centered at other locations

* Part 2: Let $f(z)$ be analytic for $|z| < 1$. If $|f(z)| \le 1$ for $|z| < 1$, and $f(0) = 0$, then $|f'(0)| \le 1$ with equality if and only if $f(z) = \lambda z$ for some constant λ with $|\lambda| = 1$

$$\int_0^{\pi} \frac{\sin^2\theta}{a+\cos\theta}\, d\theta$$