Gamelin Complex Analysis

Chapter 1: The Complex Plane and Elementary Functions

1. Complex Numbers

$$z = x + iy, \quad x, y \in \mathbb{R}$$

$$x = \operatorname{Re} z \qquad y = \operatorname{Im} z$$

Complex Plane has a one-to-one correspondence with $\mathbb{R}^2$

$$z = x + iy \longrightarrow (x, y)$$

Addition: $(x+iy)(u+iv) = (x+u) + i(y+v)$

Modulus: $|z| = \sqrt{x^2+y^2}$

Triangle Inequality: $|z+w| \leq |z| + |w|$

$z = z - w + w \rightarrow |z| \leq |z-w| + |w|$

$$|z-w| \geq |z| - |w|$$

Multiplication: $(x+iy)(u+iv) = xu - yv + i(xv + yu)$

Associative Law: $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

Commutative Law: $z_1 z_2 = z_2 z_1$

Distributive Law: $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Complex Conjugate: $\bar{z} = x - iy$

Reflection of z over the x-axis

Properties of Complex Conjugation

$$\overline{z+w} = \bar{z} + \bar{w}$$

$$\overline{zw} = \bar{z}\,\bar{w}$$

$$|z| = |\bar{z}|$$

$$|z|^2 = z\bar{z}$$

Multiplicative Inverse: $\frac{1}{z} = \frac{1}{x+iy} \times \frac{(x-iy)}{(x-iy)} = \frac{x-iy}{x^2+y^2}$

Alternatively, $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

$$\operatorname{Re}(z) = \frac{z+\bar{z}}{2} = \frac{x+iy+x-iy}{2} = \frac{2x}{2}$$

$$\operatorname{Im}(z) = \frac{z-\bar{z}}{2i} = \frac{x+iy-x+iy}{2i} = \frac{2iy}{2i}$$

Any polynomial with complex coefficients can be factored as a product of linear factors

Fundamental Theorem of Algebra: Every complex polynomial $p(z)$ of degree $n \geq 1$ has factorization

$$p(z) = c(z-z_1)^{m_1} \cdots (z-z_k)^{m_k}$$

2. Polar Representation

Points in the plane can be described by r and θ in polar coordinates

$r = \sqrt{x^2+y^2}$ and $\theta = \measuredangle((x,y), \text{x-axis})$

Conversion between polar representation and $\mathbb{R}^2$

$x = r\cos\theta$
$y = r\sin\theta$

In complex notation

$$z = x+iy = r(\cos\theta + i\sin\theta)$$

$$r = |z| \quad \text{and} \quad \theta = \arg z$$

Arg z is multivalued but its principal value satisfies $-\pi < \theta \leq \pi$

From Euler's identity

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Thus,

$z = re^{i\theta}$ $\quad r = |z|, \theta = \arg z$ ← Polar Representation of z

Useful identities

$$|e^{i\theta}| = 1$$

$\overline{e^{i\theta}} = e^{-i\theta}$ $\qquad \arg \bar{z} = -\arg z$

$1/e^{i\theta} = e^{-i\theta}$ $\qquad \arg(1/z) = -\arg z$

Addition Formula

$$e^{i(\theta+\psi)} = e^{i\theta}e^{i\psi} = (\cos\theta + i\sin\theta)(\cos\psi + i\sin\psi)$$

$\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$

expands to addition formulas for sine and cosine

de Moivre's formula

$$\cos(n\theta) + i\sin(n\theta) = e^{in\theta} = (e^{i\theta})^n = (\cos\theta + i\sin\theta)^n$$

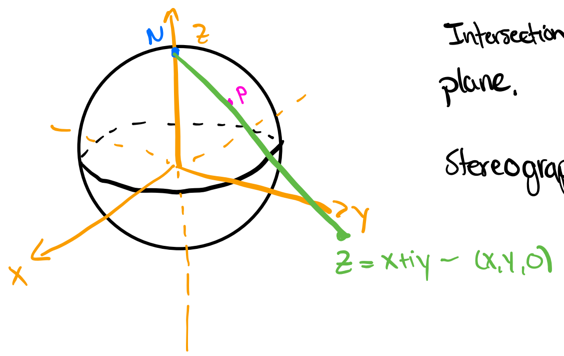
n^{th} roots of unity

$w_k = e^{2\pi i k/n}$ $\qquad 0 \leq k \leq n-1$

3. Stereographic Projection

Extended complex plane $\mathbb{C}^* = \mathbb{C}^* \cup \{\infty\}$

Stereographic projection is a way to visualize extended complex plane maps unit sphere in $\mathbb{R}^3$ to the extended complex plane



Intersection of the line between arbitrary point P and North pole and $z=0$ plane.

Stereographic projection of N is ∞

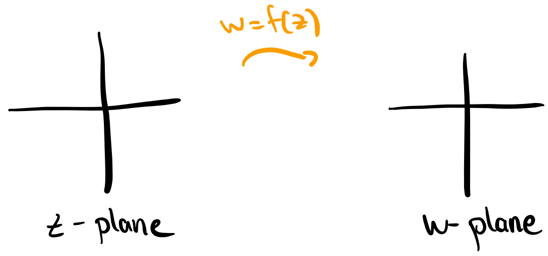
Explicit Formula

$$\begin{cases} X = 2x/(|z|^2+1) \\ Y = 2y/(|z|^2+1) \\ Z = 1 - 1/t = \dfrac{(|z|^2-1)}{|z|^2+1} \end{cases}$$

Under stereographic projection circles on the sphere correspond to circles and straight lines in the plane

4. The Square and Square Root Functions

graph functions as transformations in the complex plane



Inverse functions can hit two values so the bounds must be restricted via branches or slit planes

5. The Exponential Function

$$e^z = e^x \cos y + i e^x \sin y \qquad z = x+iy \in \mathbb{C}$$

6. The Logarithm Function

$$z \neq 0 \quad \log z = \underbrace{\log|z|}_{u} + i\,\underbrace{\arg z}_{v}$$ ← Principal value of $\log z$

7. Power Functions and Phase Factors

$$z^{\alpha} = e^{\alpha \log z} \qquad z = 0$$

Phase Change Lemma: Let $g(z)$ be a single-valued function that is defined and continuous near z_0. For any continuously varying branch of $(z-z_0)^{\alpha}$ the function $f(z) = (z-z_0)^{\alpha} g(z)$ is multiplied by the phase factor $e^{2\pi i \alpha}$ when z transverses a complete circle about z_0 in the positive direction.

8. Trigonometric and Hyperbolic Functions

$$\cosh z = \frac{e^{z} + e^{-z}}{2} \qquad \sinh z = \frac{e^{z} - e^{-z}}{2} \qquad z \in \mathbb{C}$$

Furthermore,

$$\cosh(iz) = \cos z \qquad \cos(iz) = \cosh z$$

$$\sinh(iz) = i \sin z \qquad \sin(iz) = i \sinh z$$

Chapter II: Analytic Functions

1. Review of Basic Analysis

Convergence: A sequence of complex numbers $\{s_n\}$ converges to s if for any $\varepsilon > 0$, there is an integer $N \geq 1$ such that $|s_n - s| < \varepsilon$ for all $n \geq N$

Theorem: A convergent sequence is bounded.

If $s_n \to s$ and $t_n \to t$

a) $s_n + t_n \to s + t$

b) $s_n t_n \to st$

c) $\frac{s_n}{t_n} \to s/t$ provided that $t \neq 0$

Theorem: If $r_n \leq s_n \leq t_n$ $r_n \to L$ $t_n \to L$ so $s_n \to L$

Theorem: A bounded monotone sequence of real numbers converges

Theorem: A sequence of complex numbers converges iff the corresponding real and imaginary parts converge

Theorem: A sequence of complex numbers converges iff it is a Cauchy sequence

Continuous function is continuous at each point in its domain

A subset U of the complex plane is open if $\forall z \in U$ there is a disk centered at z that is contained in U.

A subset D of the complex plane is a domain if it is open and if any two points can be connected by a broken line segment

A convex set is a set where any two points in the set can be joined by a straight line segment

A star-shaped set is a set where all points can be connected via a straight line to z_0

Convex set is star shaped with respect to all of its points

A boundary of a set E contains points z such that every disk contains points in E and not in E

A compact set is closed and bounded

Theorem: A continuous real-valued function on a compact set attains its maximum

2. Analytic Functions

Complex derivative of $f(z)$ at z_0

$$\frac{df}{dz}(z_0) = f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

Theorem: If $f(z)$ is differentiable at z_0 then $f(z)$ is continuous at z_0

Complex Derivative satisfies usual derivative rules

a) $(cf)'(z) = cf'(z)$

b) $(f+g)'(z) = f'(z) + g'(z)$

c) $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$

d) $\left(\frac{f}{g}\right)'(z) = \dfrac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$ $\quad g(z) \neq 0$

Chain Rule also holds for complex derivative

$$(f \circ g)'(z) = f'(g(z)) \cdot g'(z)$$

A function is analytic on the open set U if $f(z)$ is complex differentiable at each point of U and the complex derivative $f'(z)$ is continuous on U

3. The Cauchy Riemann Equations

Suppose $f = u + iv$ is analytic on Domain D. For a point $z \in D$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Approach Δz from x-axis (reals) and y-axis (imaginary)

Along x-axis:

$$\frac{f(z+\Delta x) - f(z)}{\Delta x} = \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - (u(x,y) + iv(x,y))}{\Delta x}$$

$$= \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + \frac{i(v(x+\Delta x, y) - v(x,y))}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Along y-axis: same logic $\Delta z = i\Delta y$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Since the derivatives must be equal

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

Cauchy Riemann Equations

Proof that partials of u, v exist, are continuous and satisfy C-R equations

Taylor's Theorem:

$$u(x+\Delta x, y+\Delta y) = u(x,y) + \frac{\partial u}{\partial x}(x,y)\Delta x + \frac{\partial u}{\partial y}(x,y)\Delta y + R(\Delta x, \Delta y)$$

error terms

$$v(x+\Delta x, y+\Delta y) = v(x,y) + \frac{\partial v}{\partial x}(x,y)\,\Delta x + \frac{\partial v}{\partial y}(x,y)\,\Delta y + S(\Delta x, \Delta y)$$

So,

$$f(z+\Delta z) = f(z) + \frac{\partial u}{\partial x}(x,y)\Delta x + \frac{\partial u}{\partial y}(x,y)\Delta y + R(\Delta z)$$
$$+ i\frac{\partial v}{\partial x}(x,y)\,\Delta x + i\frac{\partial v}{\partial y}(x,y)\,\Delta y + iS(\Delta z)$$

Substituting in C-R equations and using $\Delta z = \Delta x + i\Delta y$

$$f(z+\Delta z) = f(z) + \left(\frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y)\right)\Delta z + R(z) + iS(z)$$

$$\frac{f(z+\Delta z)}{\Delta z} = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) + \frac{R(z) + iS(z)}{\Delta z} \longrightarrow \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Theorem: If $f(z)$ is anlalytic and real-valued on domain D, then $f(z)$ is constant

$v=0$ so C-R makes $\frac{\partial u}{\partial x} = 0 \quad \frac{\partial u}{\partial y} = 0$

4. Inverse Mappings and the Jacobian

f can be considered a map from D to $\mathbb{R}^2$

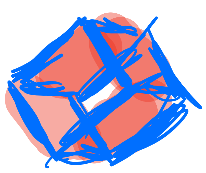
Associated Jacobian:

$$J_f = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

$$\det J_f = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \underset{\text{C-R Equations}}{=} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2$$

Theorem: If $f(z)$ is analytic, then J_f has determinant $|f'(z)|^2$

$$\det J_f = |f'(z)|^2$$



Using inverse function theorem

Theorem: Suppose $f(z)$ is analytic on domain D, $z_0 \in D$ and $f'(z_0) \neq 0$. Then there is a small disk $U \subset D$ containing z_0 such that $f(z)$ is one-to-one on U, the image $V = f(U)$ of U is open, and the inverse function

$$f^{-1} : V \to U$$

5. Harmonic Functions

$$\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0 \quad \leftarrow \text{Laplace's Equation}$$

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots \frac{\partial^2}{\partial x_n^2} \quad \leftarrow \text{laplacian operator}$$

Smooth functions that satisfy laplace's equation are harmonic

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Theorem: If $f = u + iv$ is analytic, and the functions u and v have continuous second-order partial derivatives, then u and v are harmonic

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial^2 u}{\partial y^2}$$

Same process for v

If u is harmonic on a domain D and v is harmonic function such that $u + iv$ is analytic, v is the harmonic conjugate of u

The harmonic conjugate is unique up to adding a constant

$$V(x,y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x,t)\,dt - \int_{x_0}^{x} \frac{\partial u}{\partial y}(s,y_0)\,ds + C$$

↑ when D is a rectangle, open disk or complex plane

6. Conformal Mappings

Let $\gamma(t) = x(t) + iy(t)$ be a smooth parameterized curve terminating at $\gamma(0)$

$\gamma'(0) = x'(0) - iy'(0)$ ← tangent vector to curve γ at z_0

Theorem: If $\gamma(t)$, $0 \le t \le 1$ is a smooth parameterized curve terminating at $z_0 = \gamma(0)$, and $f(z)$ is analytic at z_0 then the tangent to the curve $f(\gamma(t))$ terminating at $f(z_0)$ is:

$$(f \circ \gamma)'(0) = f'(z_0)\gamma'(0)$$

A function is conformal if it preserves angles

⋮ Skipped

7. Fractional Linear Transformations

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Chapter 3: Line Integrals and Harmonic Functions

1. Line Integrals and Green's Theorem

A path from A to B is a continuous function $t \to \gamma(t), a \le t \le b$, s.t. $\gamma(a) = A$ and $\gamma(b) = B$

A path is simple if $\gamma(s) \ne \gamma(t)$ when $s \ne t$

A closed path starts and ends at the same point

Composition of paths with the same start and end points is called a reparameterization

Trace of a path γ is the image of $\gamma[a,b]$

piecewise smooth path is a concatenation of paths with sufficient derivatives

Line Integral of $Pdx + Qdy$ along γ

$$\int_\gamma Pdx + Qdy = \int_a^b P(x(t),y(t))\frac{dx}{dt}\cdot dt + \int_a^b Q(x(t),y(t))\cdot\frac{dy}{dt}dt$$

Use this equation to calculate line integral

Line integrals are independent of parameterization

Changing direction of parameterization adds a negative sign

<u>Green's Theorem</u>: Let D be a bounded domain in the plane whose boundary ∂D consists of finite number of disjoint piecewise smooth closed curves. Let P and Q be continuously differentiable functions on $D \cup \partial D$. Then,

$$\int_{\partial D} Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Proof involves showing the relation is true for triangles and extrapolating to all domains

2. Independence of Path

Fundamental Theorem of Calculus

Part 1: $\int_a^b f(t)dt = F(b) - F(a)$ Part 2: $F(t) = \int_a^t f(s)ds \qquad a \le t \le b$

For a continuously differentiable complex valued function $h(x)$,

$$dh = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy$$

$Pdx + Qdy$ is said to be exact if $Pdx + Qdy = dh$ for some function h

Theorem (Part 1): If γ is a piecewise smooth curve from A to B and if h is continuously differentiable on γ, then

chain rule

$$\int_\gamma dh = \int_\gamma \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy = \int_a^b \frac{\partial h}{\partial x}\frac{\partial x}{\partial t}dt + \int_a^b \frac{\partial h}{\partial y}\frac{\partial y}{\partial t}dt = \int_a^b \frac{d}{dt}h(x(t),y(t))\,dt$$

$$= h(B) - h(A)$$

Exact differentials are easy to solve

$\int P\,dx + Q\,dy$ is path independent is equivalent to $\int P\,dx + Q\,dy = 0$ for any closed path

Theorem: For continuous complex-valued functions on domain D, P and Q, $\int P\,dx + Q\,dy$ is independent of path in D if and only if $P\,dx + Q\,dy$ is exact, $dh = P\,dx + Q\,dy$. The function h is unique up to adding a constant

$\Rightarrow$ Suppose $\int P\,dx + Q\,dy$ is path independent in D

For a given point $A \in D$, define $h(x,y)$ on D such that

$$h(B) = \int_A^B P\,dx + Q\,dy \qquad B \in D$$

For any point X near X_0, $h(X, Y_0)$ can be evaluated by following a path from A to (X_0, Y_0) and then following a secondary path defined by $X(t) = t$ and $Y(t) = Y_0$

Thus,

$$h(X, Y_0) = \int_\gamma P\,dx + Q\,dy + \int_{X_0}^{X} P(t, Y_0)\,dt$$

Similarly

$$h(X_0, Y) = \int_\gamma P\,dx + Q\,dy + \int_{Y_0}^{Y} Q(t, X_0)\,dt$$

Taking the derivatives via Fundamental Theorem of Calculus

$$\frac{\partial h}{\partial x}(X_0, Y_0) = P(X_0, Y_0) \qquad \text{and} \qquad \frac{\partial h}{\partial y}(X_0, Y_0) = Q(X_0, Y_0)$$

$\int_\gamma P\,dx + Q\,dy$ is constant so goes to 0

Thus, $\partial h = P\,dx + Q\,dy$

Uniqueness makes sense if you think about it

$\Leftarrow$

$$\int_\gamma P\,dx + Q\,dy = \int_\gamma dh = h(B) - h(A)$$ ← Proven above

$P\,dx + Q\,dy$ is said to be closed if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \leftarrow \text{Integrand in Green's Theorem is } 0$$

Green's Theorem implies that if $P\,dx + Q\,dy$ is closed on D, $\int_{\partial U} P\,dx + Q\,dy = 0$

Theorem: Exact differentials are closed

$$P = \frac{\partial h}{\partial x} \quad \text{and} \quad Q = \frac{\partial h}{\partial y}$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\frac{\partial h}{\partial x} = \frac{\partial}{\partial x}\frac{\partial h}{\partial y} = \frac{\partial Q}{\partial x}$$

Not every closed differential is exact

Theorem (Part II): Let P and Q be continuously differentiable complex valued functions on a domain D.

Suppose i) D is a star-shaped domain
ii) $P\,dx + Q\,dy$ is closed on D, then $P\,dx + Q\,dy$ is exact on D.

Suppose D is star-shaped with respect to $A \in D$.

$$h(B) = \int_A^B P\,dx + Q\,dy$$

Let $B = (x_0, y_0)$ and $C = (x, y_0)$ where $A, B, C \in D$

$$\int_A^B + \int_B^C + \int_C^A (P\,dx + Q\,dy) = 0$$

via green's theorem (closed)

$$\int_A^C P\,dx + Q\,dy - \int_A^B P\,dx + Q\,dy = \int_B^C P\,dx + Q\,dy$$

$$h(x, y_0) - h(x_0, y_0) = \int_{x_0}^{x} P(t, y_0)\,dt$$

FTC:

$$\frac{\partial h}{\partial x}(x_0, y_0) = P(x_0, y_0)$$

$$\frac{\partial h}{\partial y}(x_0, y_0) = Q(x_0, y_0) \quad \leftarrow \text{Same process}$$

Thus, $P\,dx + Q\,dy$ is exact

Theorem: Let D be a domain and let $\gamma_0(t)$ and $\gamma_1(t)$, $a \le t \le b$ be two paths in D from A to B. Suppose that γ_0 can be continuously deformed to γ_1

$$\int_{\gamma_0} P\,dx + Q\,dy = \int_{\gamma_1} P\,dx + Q\,dy$$

Intuition makes sense, proof requires compactness argument Franco didn't cover

Summary:

independent of path $\Leftrightarrow$ exact $\Rightarrow$ closed
Star-shaped Domains
independent of path $\Leftrightarrow$ exact $\Leftrightarrow$ closed

3. Harmonic Conjugates

If $u(x,y)$ is harmonic, then the differential $-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$ is closed

$$P = -\frac{\partial u}{\partial y} \qquad Q = \frac{\partial u}{\partial x}$$

$$\frac{\partial P}{\partial y} = \frac{-\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial x^2} = \frac{\partial Q}{\partial x}$$

Laplace

Theorem: Any harmonic function $u(x,y)$ on a star-shaped domain D has a harmonic conjugate function $v(x,y)$ on D

Since $-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$ is closed we can use the harmonic conjugate formula from section 2

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

Deriving with respect to dx and dy we find the C-R equations

Thus $u+iv$ is analytic

$$v(B) = \int_A^B -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy$$

A is fixed and the integral is path independent

4. The Mean Value Property

Let $h(z)$ be a continuous real valued function on domain D. Let $z_0 \in D$ and suppose D contains the disk $\{|z - z_0| < \rho\}$

Average value of $h(z)$ on the circle $\{|z - z_0| = r\}$

$$A(r) = \int_0^{2\pi} h(z_0 + re^{i\theta})\,\frac{d\theta}{2\pi} \qquad 0 < r < \rho$$

$A(r) \to h(z_0)$ as r decreases to 0

Theorem: If $u(z)$ is a harmonic function on a domain D, and if the disk $\{|z - z_0| < \rho\}$ is contained in D, then

$$u(z_0) = \int_0^{2\pi} u(z_0 + re^{i\theta})\,\frac{d\theta}{2\pi} \qquad 0 < r < \rho$$

Average value of a harmonic function on the boundary of a disk is its value at the center

$$0 = \oint_{|z - z_0| = r} -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy$$

closed so green's makes 0

parameterizing as a circle

$$0 = r\int_0^{2\pi}\left[\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right]d\theta = r\int_0^{2\pi}\frac{\partial u}{\partial r}(z_0 + re^{i\theta})\,d\theta$$

Because $u(z)$ is smooth you can interchange order of integration/differentiation

After dividing by $2\pi r$

$$0 = \frac{\partial}{\partial r}\int_0^{2\pi} u(z_0 + re^{i\theta})\,\frac{d\theta}{2\pi}$$

Thus $\displaystyle\int_0^{2\pi} u(z_0 + re^{i\theta})\,\frac{d\theta}{2\pi}$ is constant for $0 < r < \rho$

Limit as $r \to 0$ tells us that this constant is $u(z_0)$

Harmonic Functions have the mean value property

5. The Maximum Principle

Strict Maximum Principle (Real Version): Let $u(z)$ be a real valued harmonic function on domain D s.t. $u(z) \leq M$ $\forall z \in D$. If $u(z_0) = M$ for some $z_0 \in D$ then $u(z) = M$ $\forall z \in D$

Suppose $u(z_1) = M$

$$0 = \int_0^{2\pi} u(z_1) - u(z_1 + re^{i\theta}) \frac{d\theta}{2\pi} \qquad 0 < r < \rho$$

Since integrand is non negative and continuous this can only be true if the integrand is 0.

$$u(z_1) = u(z_1 + re^{i\theta}) = M$$

Thus, There exists a disk centered around each point in the set $\{u(z) = M\}$. Thus, it is open.

The set $\{u(z) < M\}$ is also open since $u(z)$ is continuous.

Since D is a domain $\{u(z) < M\}$ or $\{u(z) = M\}$ is D and the other is empty.

Thus we have the strict maximum principle.

Strict Maximum Principle (Complex Version): Let h be a bounded complex-valued harmonic function on a domain D. If $|h(z)| \leq M$ for all $z \in D$, and $|h(z_0)| = M$ for some $z_0 \in D$ then $h(z_0)$ is constant on D.

Proof follows from real version (See Gamelin page 88)

Maximum Principle: Let $h(z)$ be a complex valued harmonic function on a bounded domain D Such that $h(z)$ extends continuously to the boundary ∂D of D. If $|h(z)| \leq M$ $\forall z \in \partial D$, then $|h(z)| \leq M$ for all $z \in D$

If it reaches max it will be constant

Chapter 4: Complex Integration and Analyticity

1. Complex Line Integrals

Define $dz = dx + i\,dy$

Then, $$\int_\gamma h(z)\,dz = \int_\gamma h(z)\,dx + i \int_\gamma h(z)\,dy$$

If γ is parameterized by $t \to z(t) = x(t) + i\,y(t)$, then the Riemann sum approximation gives us

$$\int_\gamma h(z)\,dz \approx \sum h(z_j)(z_{j+1} - z_j)$$

Length of a path γ

$$L = \int_\gamma |dz| = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$|dz| = ds = \sqrt{(dx)^2 + (dy)^2}$$

$$L \approx \sum |z_{j+1} - z_j|$$

Theorem: Suppose γ is a piecewise smooth curve. If $h(z)$ is a continuous function on γ then,

$$\left| \int_\gamma h(z)\,dz \right| \le \int_\gamma |h(z)|\,|dz|$$

$\to$ triangle inequality for integrals

If γ has length L and $|h(z)| \le M$ on γ

$$\left| \int_\gamma h(z)\,dz \right| \le ML$$

An estimate is considered sharp if it cannot improve.

Estimates can sometimes be improved by considering parameterizations

2. Fundamental Theorem of Calculus for Analytic Functions

For a continuous function $f(z)$ on domain D, $F(z)$ is said to be primitive if $F(z)$ is analytic and $F'(z) = f(z)$

Theorem (Part 1): If $f(z)$ is continuous on domain D, and if $F(z)$ is primitive for $f(z)$

$$\int_A^B f(z)\,dz = F(B) - F(A)$$

$$F'(z) = \frac{\partial F}{\partial x} = \frac{1}{i}\frac{\partial F}{\partial y}$$

$$F(B) - F(A) = \int_A^B dF = \int_A^B \frac{\partial F}{\partial x}dx + \frac{1}{i}\frac{\partial F}{\partial y}dy = \int_A^B F'(z)(dx + i\,dy) = \int_A^B F'(z)\,dz$$

To integrate curves that are undefined at certain points, one can creatively redefine the path so that the integral can still be evaluated.

Theorem (Part 2): Let D be a star-shaped domain, and let $f(z)$ be analytic on D. Then $f(z)$ has a primitive on D and the primitive is unique up to adding a constant. A primitive for $f(z)$ is given by

$$F(z) = \int_{z_0}^{z} f(\zeta)\,d\zeta \qquad z \in D$$

z_0 is a fixed point in D

Proof: $f = u + iv$

Via C-R equations we know that $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Thus we know that $u\,dx - v\,dy$ is closed and exact

Since $u\,dx - v\,dy$ is exact we know that there exists a continuously differentiable function U on D s.t.

$$dU = u\,dx - v\,dy$$

$$\frac{\partial U}{\partial x} = u \quad , \quad \frac{\partial U}{\partial y} = -v$$

$$\frac{\partial U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \qquad \Rightarrow U \text{ is harmonic}$$

C-R equations

Since U is harmonic on a star-shaped domain, there is a conjugate harmonic function V for U on D.

$G = U + iV$ is analytic on D.

$$G' = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = u + iv = f.$$

$\therefore$ G is primitive for $f(z)$

3. Cauchy's Theorem

Theorem: A continuously differentiable function $f(z)$ on D is analytic if and only if the differential $f(z)dz$ is closed

$$f(z) = u + iv$$

$$f(z)dz = u+iv\,(dx + i\,dy) = (u+iv)dx + (-v+ui)dy$$

Using C-R we see

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

So,

$$\frac{\partial}{\partial y}(u+iv) = \frac{\partial}{\partial x}(-v+iu) \Rightarrow f(z)dz \text{ is closed}$$

Cauchy's Theorem: Let D be a bounded domain with piecewise smooth boundary. If $f(z)$ is analytic on D that extends smoothly to ∂D, then

$$\int_{\partial D} f(z)dz = 0$$

$\leftarrow$ direct application of Green's Theorem

$$\int_{\partial D} Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

4. Cauchy Integral Formula

Cauchy Integral Formula: Let D be a bounded domain with piecewise smooth boundary. If $f(z)$ is analytic on D, and $f(z)$ extends smoothly to the boundary of D, then

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}\,dw \quad , \quad z \in D$$

Proof: fix a point $z \in D$ and for small $\varepsilon > 0$, let $D_\varepsilon = D \setminus \{|w-z| \le \varepsilon\}$ where D_ε is obtained from D by punching out a disk of radius ε centered at z

∂D_ε is the union of ∂D and the circle $\{|w-z| = \varepsilon\}$

$\frac{f(w)}{(w-z)}$ is analytic for $w \in D_\varepsilon$ so by Cauchy's Theorem,

$$\int_{\partial D_\varepsilon} \frac{f(w)}{w-z}\,dw = 0$$

Reversing the orientation of circle changes the sign of the integral

$$\int_{\partial D_\varepsilon} \frac{f(w)}{w-z}\,dw = \int_{|w-z|=\varepsilon} \frac{f(w)}{w-z}\,dz + \int_{\partial D} \frac{f(w)}{w-z}\,dz = -\underbrace{\int_{|w-z|=\varepsilon} \frac{f(w)}{w-z}\,dz}_{\text{reversing orientation}} + \int_{\partial D} \frac{f(w)}{w-z}\,dz = 0$$

$$\int_{|w-z|=\varepsilon} \frac{f(w)}{w-z}\,dz = \int_{\partial D} \frac{f(w)}{w-z}\,dz$$

writing $w = z + \varepsilon e^{i\theta}$ and $dw = i\varepsilon e^{i\theta}$ and dividing both sides by $2\pi i$

$$\int_0^{2\pi} f(z+\varepsilon e^{i\theta}) \frac{d\theta}{2\pi} = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w-z}\, dw$$

$$f(z) = \int_0^{2\pi} f(z+\varepsilon e^{i\theta}) \frac{d\theta}{2\pi} = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w-z}\, dw$$

mean value property

Can also be argued by taking the limit as $\varepsilon \to 0$ and noticing that the LHS integral is the average value in the circle around z.

Theorem: Let D be a bounded domain with piecewise smooth boundary. If $f(z)$ is an analytic function on D that extends smoothly to the boundary of D, then $f(z)$ has complex derivatives of all orders on D, which are given by

$$f^{(m)}(z) = \frac{m!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{m+1}}\, dw \qquad z \in D,\ m \geq 0$$

Proof:

$$= \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{1}{\Delta z}\left[\frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-(z+\Delta z)}\, dw - \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}\, dz\right]$$

$$= \frac{1}{2\pi i}\int_{\partial D} f(w) \cdot \frac{1}{(w-(z+\Delta z))(w-z)}\, dw$$

As $z \to 0$ the integrand approaches $\dfrac{f(w)}{(w-z)^2}$

$$f'(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z)^2}\, dw$$

That proves $m=1$

For $m>1$ prove via induction

$$(w-z-\Delta z)^m = (w-z)^m - m(w-z)^{m-1}\Delta z + \frac{m(m-1)}{2}(w-z)^{m-2}(\Delta z)^2 + \cdots$$

binomial expansion

$$\frac{1}{(w-(z+\Delta z))^m} - \frac{1}{(w-z)^m} = \frac{(w-z)^m - (w-z-\Delta z)^m}{(w-z)^m (w-z-\Delta z)^m} = \frac{m\Delta z}{(w-z)(w-z-\Delta z)^m} - \frac{m(m-1)(\Delta z)^2}{2(w-z)(w-z-\Delta z)^m} + \cdots$$

$$\frac{(m-1)!}{2\pi i}\int_{\partial D} f(w)\left[\frac{m}{(w-z)(w-z-\Delta z)^m} + \Delta z(\cdots)\right]dw$$

Corollary: If $f(z)$ is analytic on domain D, then $f(z)$ is infinitely differentiable and each complex derivative is analytic on D.

Integrals that cannot be evaluated in terms of CIF can be evaluated by punching holes in the domain and using Cauchy's formula

see page 116

5. Liouville's Theorem

Cauchy Estimates: Suppose $f(z)$ is analytic for $|z-z_0|\le \rho$. If $|f(z)|\le M$ for $|z-z_0|=\rho$ then

$$|f^{(m)}(z_0)| \le \frac{m!}{\rho^m}M$$

Proof: $$f^{(m)}(z_0) = \frac{m!}{2\pi i}\int_{|z-z_0|=\rho}\frac{f(z)}{(z-z_0)^{m+1}}dz$$

Parameterize: $z = z_0 + \rho e^{i\theta}$ $\quad dz = i\rho e^{i\theta}d\theta$

$$\frac{1}{2\pi i} \frac{f(z)}{(z-z_0)^{m+1}} dz = \frac{f(z_0+\rho e^{i\theta})}{\rho^m e^{im\theta}} \frac{d\theta}{2\pi}$$

$$f^{(m)}(z_0) = \frac{m!}{\rho^m} \int_0^{2\pi} f(z_0+\rho e^{i\theta}) e^{-im\theta} \frac{d\theta}{2\pi}$$

$$|f^{(m)}(z_0)| \le \frac{m!}{\rho^m} \int_0^{2\pi} |f(z_0+\rho e^{i\theta})| \frac{d\theta}{2\pi}$$

Liouville's Theorem: Let $f(z)$ be an analytic function on the complex plane. If $f(z)$ is bounded, then $f(z)$ is constant

Suppose $|f(z)| \le M \quad \forall z \in \mathbb{C}$.

Applying cauchy estimate we get

$$|f'(z_0)| \le \frac{M}{\rho}$$ for any z_0 and ρ (arbitrary disk centered at z_0 with radius ρ)

As $\rho \to \infty$

$|f'(z_0)| \le 0$ and so we find that $f'(z_0) = 0$

Since this is true for all z_0, $f(z)$ is constant.

An entire function is analytic on the entire complex plane

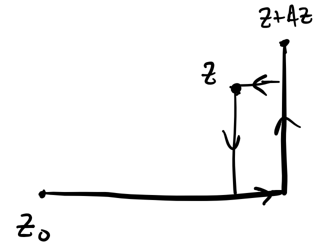
$1/z$, $\log z$, and $\sqrt{z}$ are not entire

A bounded entire function is constant $\rightarrow$ Liouville's

6. Morera's Theorem

Morera's Theorem: Let $f(z)$ be a continuous function on a domain D. If $\int_{\partial R} f(z)dz = 0$ for every closed rectangle R contained in D with sides parallel to the coordinate axis, then $f(z)$ is analytic on D.

★ Proof: Assume D is a disk with center z_0



Define

$$F(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta \qquad z \in D$$

where the path of integration is horizontal, then vertical.

$$F(z+\Delta z) - F(z) = \int_{z}^{z+\Delta z} f(\zeta)\, d\zeta$$

Since $f(z)$ is constant

$$F(z+\Delta z) - F(z) = f(z)\int_{z}^{z+\Delta z} d\zeta + \int_{z}^{z+\Delta z} (f(\zeta) - f(z))\, d\zeta = f(z)\,\Delta z + \int_{z}^{z+\Delta z} (f(\zeta) - f(z))\, d\zeta$$

Length from z to $z+\Delta z$ is at most $2|\Delta z|$

$$\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| \le 2M_\varepsilon$$

ML estimate

M_ε is the maximum of $|f(\zeta) - f(z)|$ where $|\zeta - z| \le \varepsilon$

$M_\varepsilon \to 0$ as $\varepsilon \to 0$

Therefore,

$$F'(z) = f(z)$$

$f(z)$ continuous $\to$ $F(z)$ analytic $\to$ $f(z)$ analytic

7. Goursat's Theorem

Goursat's Theorem: If $f(z)$ is a complex-valued function on a domain D s.t.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$ exists at each point z_0 of D, then $f(z)$ is analytic on D.

Proof: Let R be a closed rectangle in D. Subdivide R into four equal subrectangles.

$$\left|\int_{\partial R_1} f(z)\,dz\right| \geq \frac{1}{4}\left|\int_{\partial R} f(z)\,dz\right| \qquad \text{since} \quad \sum_{i=1}^{4} \int_{\partial R_i} f(z)\,dz = \int_{\partial R} f(z)\,dz$$

Repeating this process ...

$$\left|\int_{\partial R_n} f(z)\,dz\right| \geq \frac{1}{4}\left|\int_{\partial R_{n-1}} f(z)\,dz\right| \geq \cdots \geq \frac{1}{4^n}\left|\int_{\partial R} f(z)\,dz\right|$$

R_n are decreasing and approach $z_0 \in D$

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| \leq \varepsilon_n \qquad z \in R_n$$

$$\varepsilon_n \to 0 \quad \text{as} \quad n \to \infty$$

If L is the length of ∂R then the length of $\partial R_n = \frac{L}{2^n}$

$$\left|f(z)-f(z_0) - f'(z_0)(z-z_0)\right| \leq \varepsilon_n |z-z_0| \leq 2\varepsilon_n \frac{L}{2^n}$$

Using ML estimate

$$\left|\int_{\partial R_n} f(z)\,dz\right| = \left|\int_{\partial R_n} [f(z)-f(z_0)-f'(z-z_0)]\,dz\right| \leq 2\varepsilon_n \frac{L}{2^n} \cdot \frac{L}{2^n} = \frac{2L^2\varepsilon_n}{4}$$

?

$$\left|\int_{\partial R} f(z)\,dz\right| \leq 4^n \left|\int_{\partial R_n} f(z)\,dz\right| \leq 2L^2 \varepsilon_n \to 0$$

By morera's theorem, $f(z)$ is analytic

8. Complex Notation and Pompeiu's Formula

Not covered in class

Chapter 5: Power Series

1. Infinite Series

A series $\sum_{k=0}^{\infty} a_k$ of complex numbers converges to S if the sequence of partial sums $S_k = a_0 + \dots + a_k$ converges to S

Comparison Test: If $0 \le a_k \le r_k$ and if $\sum r_k$ converges, then $\sum a_k$ converges and $\sum a_k \le \sum r_k$

Theorem: If $\sum a_k$ converges, then $a_k \to 0$ as $k \to \infty$

Geometric Sum

$$\sum_{k=0}^{\infty} z^k \quad , \quad S_k = \frac{1 - z^{k+1}}{1-z} \qquad z \neq 1$$

If $|z| < 1$

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$$

A series converges absolutely if $\sum |a_k|$ converges

Theorem: If $\sum a_k$ converges absolutely, then $\sum a_k$ converges

$$\left| \sum_{k=0}^{\infty} a_k \right| \le \sum_{k=0}^{\infty} |a_k|$$

For geometric series

$$\frac{1}{1-z} - \sum_{k=0}^{n} z^k = \sum_{k=n+1}^{\infty} z^k = z^{n+1} \sum_{j=0}^{\infty} z^j = \frac{z^{n+1}}{1-z}$$

$$\left| \frac{1}{1-z} - \sum_{k=0}^{n} z^k \right| \le \frac{|z|^{n+1}}{1-|z|} \qquad |z| < 1$$

2. Sequences and Series of Functions

Let $\{f_j\}$ be a sequence of complex-valued functions defined on a set E. $\{f_j\}$ converges pointwise on E if for each point $x \in E$, $\{f_j(x)\}$ converges. The limit $f(x)$ of $\{f_j(x)\}$ is the complex valued function on E

pointwise limit of a series of continuous functions need not be continuous

The sequence $\{f_j\}$ of functions on E converges uniformly to f on E if $|f_j(x) - f(x)| \le \epsilon_j \quad \forall x \in E$ where $\epsilon_j \to 0$ as $j \to \infty$

$$\epsilon_j = \sup_{x \in E} |f_j(x) - f(x)|$$

Uniform convergence is stronger than pointwise convergence

Theorem: Let $\{f_j\}$ be a sequence of complex-valued functions defined on a subset E of the complex plane. If each f_j is continuous on E and if $\{f_j\}$ converges uniformly to f on E, then f is continuous on E.

Theorem: Let γ be a piecewise smooth curve in the complex plane. If $\{f_j\}$ is a sequence of continuous complex-valued functions on γ and if $\{f_j\}$ converges uniformly to f on γ, then $\int_\gamma f_j(z)dz$ converges to $\int_\gamma f(z)dz$

Let ϵ_j be the worst case estimator for $f_j - f$ on γ and L be the length of γ

$$|f_j - f| \leq \epsilon_j$$

$$\left| \int_\gamma f_j(z) - \int_\gamma f(z)dz \right| \leq \epsilon_j L$$

ML estimate

This tends to 0 so $\int_\gamma f_j(z)dz \to \int_\gamma f(z)dz$

Let $\sum g_j(x)$ be a series of complex valued functions defined on E

$$S_n(x) = \sum_{k=0}^{n} g_k(x) = g_0(x) + g_1(x) + \dots g_n(x)$$

The series converges pointwise on E if the sequence of partial sums converge pointwise on E

Series converges uniformly if the sequence of partial sums converges uniformly on E

Weierstrauss M-Test: Suppose $M_k \geq 0$ and $\sum M_k$ converges. If $g_k(x)$ are complex valued functions on a set E such that $|g_k(x)| \leq M_k$ for all $x \in E$, then $\sum g_k(x)$ converges uniformly on E.

For a fixed x, $\sum g_k(x)$ is absolutely convergent and we know that $\sum |g_k(x)| \leq \sum M_k$

$\sum g_k(x)$ converges to $g(x)$

$$|g(x)| \leq \sum |g_k(x)| \leq \sum M_k$$

$$|g(x) - S_n(x)| = \left| \sum_{k=n+1}^{\infty} g_k(x) \right| \leq \sum_{k=n+1}^{\infty} M_k$$

$\epsilon_n = \sum_{k=n+1}^{\infty} M_k$ so $\epsilon_n \to 0$ as $n \to \infty$ and thus $S_n(x)$ converges to $g(x)$

Theorem: If $\{f_k(z)\}$ is a sequence of analytic functions on a domain D that converges uniformly to $f(z)$ on D, then $f(z)$ is analytic on D

Let E be a closed rectangle contained in D. By Cauchy's theorem, $\int_{\partial E} f_k(z)dz = 0$ for every k. Thus, as proven earlier, $\int_{\partial E} f(z)dz = 0$. Then, applying Morera's theorem we get that f is analytic.

Theorem: Suppose that $f_k(z)$ is analytic for $|z-z_0| \le R$ and suppose that the sequence $\{f_k(z)\}$ converges uniformly to $f(z)$ for $|z-z_0| \le R$. Then for each $r < R$ and for each $m \ge 1$, the sequence of m^{th} derivatives $\{f_k^{(m)}(z)\}$ converges uniformly to $f^{(m)}(z)$ for $|z-z_0| \le r$

Suppose $\varepsilon_k \to 0$ are s.t. $|f_k(z) - f(z)| < \varepsilon_k$ for $|z-z_0| < R$. For s s.t. $r < s < R$ The Cauchy integral formula gives us that the m^{th} derivative $f_k(z) - f(z)$ on the disk $|z-z_0| \le s$

$$f_k^{(m)}(z) - f^{(m)}(z) = \frac{m!}{2\pi i} \int_{|z-z_0|=s} \frac{f_k(\zeta) - f(\zeta)}{(\zeta - z)^{m+1}} \, d\zeta \qquad |z-z_0| \le r$$

If $|\zeta - z_0| = s$ and $|z-z_0| \le r$, then $|\zeta - z| \ge s-r$

$$\left| \frac{f_k(\zeta) - f(\zeta)}{(\zeta - z)^{m+1}} \right| \le \frac{\varepsilon_k}{(s-r)^{m+1}}$$

Using ML-estimate

$$\left| f_k^{(m)}(z) - f^{(m)}(z) \right| \le \frac{m!}{2\pi} \frac{\varepsilon_k}{(s-r)^{m+1}} 2\pi s = P_k \qquad |z-z_0| \le r$$

$$P_k \to 0 \quad \text{as} \quad k \to \infty$$

A sequence $\{f_k(z)\}$ of analytic functions on a domain D converges normally to the analytic function $f(z)$ on D if it converges uniformly to $f(z)$ on each closed disk contained in D

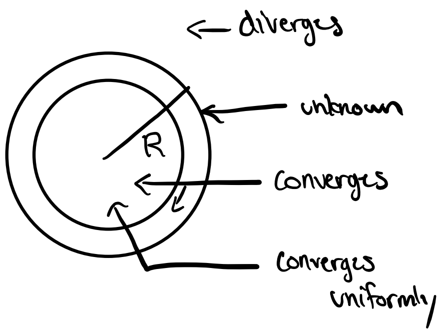
Theorem: Suppose that $\{f_k(z)\}$ is a sequence of analytic functions on a domain D that converges normally on D to the analytic function $f(z)$. Then for each $m \ge 1$ the sequence of m^{th} derivatives $\{f_k^{(m)}(z)\}$ converges normally to $f^{(m)}(z)$ on D.

dilating disks

3. Power Series

A power series centered at z_0 is a series of the form $\sum_{k=0}^{\infty} a_k (z-z_0)^k$

Theorem: Let $\sum a_k z^k$ be a powerseries. Then there is R, $0 \le R \le \infty$ s.t. $\sum a^k z^k$ converges absolutely if $|z| < R$ and $\sum a_k z^k$ doesn't converge if $|z| > R$. For each fixed r satisfying $r < R$, the series $\sum a_k z^k$ converges uniformly for $|z| \le r$.



R is the radius of convergence of the series $\sum a_k z^k$

Only dependent on the tail of the series

$|a_k| r^k$ is bounded for some $r = r_0$

Choose s s.t. $r < s < R$. $|a_k| s^k$ is bounded, $|a_k| s^k \le C$ for $k \ge 0$

If $|z| \le r$

$$|a_k z^k| \le |a_k| r^k = |a_k| s^k \left(\tfrac{r}{s}\right)^k \le C \left(\tfrac{r}{s}\right)^k$$

Let $M_k = C\left(\tfrac{r}{s}\right)^k$. Since $\sum M_k$ converges, the Weirestrauss M-test applies and $\sum a_k z^k$ converges uniformly for $|z| \le r$ and absolutely for each z.

Theorem: Suppose $\sum a_k z^k$ is a power series with radius of convergence $R > 0$. Then the function

$f(z) = \sum_{k=0}^{\infty} a_k z^k$, $|z| < R$ is analytic. The derivatives of $f(z)$ are obtained by

differentiating the series term by term

$$f'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1} \quad \ldots$$

$$a_k = \frac{1}{k!} f^{(k)}(0), \quad k \geq 0$$

Ratio Test

Theorem: If $\left|\frac{a_k}{a_{k+1}}\right|$ has a limit as $k \to \infty$, either finite or ∞, then the limit is the radius of convergence R of $\sum a_k z^k$

$$R = \lim_{k \to \infty} \left|\frac{a_k}{a_{k+1}}\right|$$

Let $L = \lim \left|\frac{a_k}{a_{k+1}}\right|$. If $r < L$, then $\left|\frac{a_k}{a_{k+1}}\right| > r$ eventually for all $k \geq N$.

$$|a_k| > r|a_{k+1}| \quad \text{for } k \geq N$$

$$|a_N| r^N \geq |a_{N+1}| r^{N+1} \ldots$$

Thus the sequence $|a_k| r^k$ is bounded.

Since $r < L$ is arbitrary, $L \leq R$

Consider $s > L$. Then $\left|\frac{a_k}{a_{k+1}}\right| < s$ eventually for $k \geq N$.

$$|a_k| < s|a_{k+1}| \quad \text{for } k \geq N$$

$$|a_N| s^N \leq |a_{N+1}| s^{N+1} \leq \ldots$$

$|a_k| s^k$ does not converge to 0 for $|z| \geq s$, so $s \geq R$

$s > L$ is arbitrary so $L \geq R$

Thus, $L = R$ □

Root test

Theorem: If $\sqrt[k]{|a_k|}$ has a limit as $k \to \infty$, either finite or $+\infty$, then radius of convergence of $\sum a_k z^k$ is given by

$$R = \frac{1}{\lim \sqrt[k]{|a_k|}}$$

If $r > \frac{1}{\lim \sqrt[k]{|a_k|}}$, then $\sqrt[k]{|a_k|}\, r > 1$ so $|a_k| r^k > 1$. The terms of the series $\sum a_k z^k$ do not converge to 0 for $|z| = r$.

If $r < \frac{1}{\lim \sqrt[k]{|a_k|}}$, then $\sqrt[k]{|a_k|}\, r < 1$ so the sequence $|a_k| r^k < 1$ is bounded.

$r \leq R$

Cauchy - Hammard Formula:

$$R = \frac{1}{\limsup \sqrt[k]{|a_k|}} \nearrow \frac{1}{\limsup |a_k| r^k}$$

where lim sup is the value where only finitely many entries are greater but infinitely many are less than S.

4. Power Series Expansion of an Analytic Function

Theorem: Suppose $f(z)$ is analytic for $|z-z_0| < \rho$. Then $f(z)$ is represented by the power series

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k, \quad |z-z_0| < \rho$$

where

$$a_k = \frac{f^{(k)}(z_0)}{k!} \qquad k \geq 0$$

where $R \geq \rho$

$$a_k = \frac{1}{2\pi i} \oint_{|\varsigma - z_0| = r} \frac{f(\varsigma)}{(\varsigma - z_0)^{k+1}} \, d\varsigma$$, for fixed r, $0 \leq r \leq \rho$ and $k \geq 0$

If $|f(z)| \leq M$ for $|z-z_0| = r$

$$|a_k| \leq \frac{M}{r^k} \qquad k \geq 0$$

Proof: For a fixed z, $|z| < r$ and $|\varsigma| = r$

$$\frac{1}{\varsigma - z} = \frac{1}{\varsigma} \cdot \frac{1}{1 - z/\varsigma} = \frac{1}{\varsigma} \sum_{k=0}^{\infty} \left(\frac{z}{\varsigma}\right)^k = \sum_{k=0}^{\infty} \frac{z^k}{\varsigma^{k+1}}$$

Series converges uniformly when $|\varsigma| = r$

$$f(z) = \frac{1}{2\pi i} \int_{|\varsigma|=r} \frac{f(\varsigma)}{\varsigma - z} d\varsigma = \frac{1}{2\pi i} \int_{|\varsigma|=r} \left(\sum f(\varsigma) \frac{z^k}{\varsigma^{k+1}} \right) d\varsigma = \frac{1}{2\pi i} \sum \left(\int_{|\varsigma|=r} \frac{f(\varsigma)}{\varsigma^{k+1}} d\varsigma \right) z^k$$

$$= \sum a_k z^k$$

Corollary: Suppose that $f(z)$ and $g(z)$ are analytic for $|z-z_0| < r$. If $f^{(k)}(z_0) = g^{(k)}(z_0)$ for $k \geq 0$ then $f(z) = g(z)$ for $|z-z_0| < r$

Corollary: Suppose $f(z)$ is analytic at z_0, with power series expansion $f(z) = \sum a_k (z-z_0)^k$ centered at z_0. Then the radius of convergence of the power series is the largest number R s.t. $f(z)$ extends to be analytic on $\{|z-z_0| < R\}$

5. Power Series Expansion at Infinity

A function $f(z)$ is said to be analytic at $z=\infty$ if the function $g(w)=f(1/w)$ is analytic at $w=0$

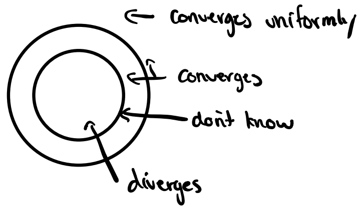
If $f(z)$ is analytic at ∞ then $g(w)$ has a power series expansion at $w=0$

$$g(w)=\sum_{k=0}^{\infty} b_k w^k = b_0 + b_1 w + \dots \qquad |w|<\rho$$

Thus,

$$f(z)=\sum_{k=0}^{\infty} \frac{b_k}{z^k} = b_0 + \frac{b_1}{z} + \dots \qquad |z|>1/\rho$$

$f(z)$ converges absolutely for $|z|>1/\rho$ and uniformly for any $r>1/\rho$ when $|z|\geq r$



To calculate coefficients

$$\int_{|z|=r} f(z)\, z^m = \int_{|z|=r} \left(\sum b_k z^{-k}\right) z^m\, dz = \sum b_k \int_{|z|=r} z^{m-k}\, dz = 2\pi i\, b_{m+1}$$

$$b_k = \frac{1}{2\pi i} \int_{|z|=r} f(z) z^{k-1}\, dz$$

Review example on 150

6. Manipulation of Power Series

If $f(z)=\sum_{k=0}^{\infty} a_k z^k$ and $g(z)=\sum_{k=0}^{\infty} b_k z^k$ are analytic at 0, $f(z)+g(z)=\sum_{k=0}^{\infty}(a_k+b_k)z^k$

power series representation

For c a complex constant

$$cf(z)=\sum_{k=0}^{\infty} c a_k z^k$$

Expand as geometric series where possible

$$f(z)g(z)=\sum_{k=0}^{\infty} c_k z^k \qquad c_k = a_k b_0 + a_{k-1} b_1 + \dots a_1 b_{k-1} + a_0 b_k$$

7. The Zeros of an Analytic Function

Let $f(z)$ be analytic at z_0 and suppose $f(z_0)=0$ but $f(z)\neq 0$.

$f(z)$ has zero of order N at z_0 if $f(z_0)=f'(z_0)=\cdots f^{(N-1)}(z_0)=0$ but $f^{(N)}(z_0)\neq 0$

$$f(z)=\sum_{k=N}^{\infty} a_k(z-z_0)^k = (z-z_0)^N h(z)$$

$h(z_0)=a_N\neq 0$ and $h(z)$ is analytic at z_0

Order of 0 is power of first term in power series

Order of Zero for $f(z)g(z)$ is sum of orders of zero at that point

A point $z_0\in E$ is an isolated point of the set E if there is $\rho>0$ s.t. $|z-z_0|\geq\rho$ for all points $z\in E$ except z_0

Theorem: If D is a domain and $f(z)$ is an analytic function on D that is not identically zero, then zeros of $f(z)$ are isolated.

Identity Principle

Connected Argument:

Let U be the set of $z\in D$ s.t. $f^{(m)}(z)=0$ for all $m\geq 0$. If $z_0\in U$ then the power series expansion of f simplifies to $f(z)=0$ for z in a disk centered at z_0. Since each point in said disk exists in U we find that U is an open set.

For $z_0\in D\setminus U$ we find that $f^{(k)}(z_0)\neq 0$ for a disk centered around z_0 in $D\setminus U$. Using similar reasoning we find that $D\setminus U$ is also open.

Since D is a domain and we have two open sets, $U=D$ or U must be null. Since f is not identically 0 we find that U is empty and zeros of $f(z)$ have finite order.

If $f(z_0)$ is a 0 of order N we find that, via the power series, $f(z)=(z-z_0)^N h(z)$ where $h(z)$ is analytic at z_0 and $h(z_0)\neq 0$. For small ρ we get that $h(z)\neq 0$ in $|z-z_0|<\rho$. Therefore, $f(z)\neq 0$ for $|z-z_0|<\rho$. Since each 0 has a distance ρ we find that the zeros of $f(z)$ are isolated.

Theorem (Uniqueness Principle): If $f(z)$ and $g(z)$ are analytic on a domain D, and if $f(z)=g(z)$ for all z belonging to a set that has a nonisolated point, then $f(z)=g(z)$ for all $z\in D$.

Direct application of identity principle with $f(z)-g(z)$

Principle of Permanence of Functional Equations: Let D be a domain and let E be a subset of D that has a nonisolated point. Let $F(z,w)$ be a function defined for $z,w\in D$ s.t. $F(z,w)$ is analytic in z for each fixed $w\in D$ and analytic for each fixed $z\in D$. If $F(z,w)=0$ whenever z and w both belong to E, then $F(z,w)=0$ for all $z,w\in D$

Double application of uniqueness principle

8. Analytic Continuation

Lemma: Suppose D is a disk, $f(z)$ is analytic on D, and $R(z_i)$ is the radius convergence of the power series expansion of $f(z)$ about a point $z_i\in D$. Then:

$$|R(z_1)-R(z_2)|\leq|z_1-z_2|$$

$R(z_1)$ is the largest disk centered at z_1 to which $f(z)$ extends analytically

$$R(z_2)\leq R(z_1)+|z_2-z_1| \qquad R(z_1)\leq R(z_2)+|z_2-z_1|$$

$$\Downarrow$$

$$|R(z_1)-R(z_2)|\leq|z_2-z_1|$$

For a path $\gamma(t)$ starting at $z_0=\gamma(a)$ and $\sum a_n(z-z_0)^n$ power series representation of $f(z)$

$f(z)$ is analytically continuable along γ if for each t $\quad a\leq t\leq b$

$$f_t(z)=\sum_{n=0}^{\infty}a_n(t)(z-\gamma(t))^n \qquad |z-\gamma(t)|<r(t)$$

$f_s(z)=f_t(z)$ for s near t when z is in the intersection of disks of convergence

Uniqueness principle tells us that $f_t(z)$ determines $f_s(z)$ and by extension $f_b(z)$ is uniquely determined by $f_a(z)$

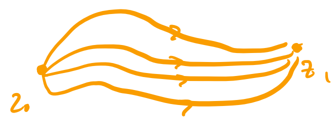
$f_b(z)$ is the analytic continuation of $f(z)$ along γ

$\hookrightarrow$ power series or analytic function near $\gamma(b)$

Theorem: Suppose $f(z)$ can be continued analytically along the path $\gamma(t)$, $a \le t \le b$. Then the analytic continuation is unique. Further for each $n \ge 0$ the coefficient $a_n(t)$ of the series depends continuously on t and the radius of convergence depends continuously on t.

Lemma: Let f, γ be as previously defined. $\delta > 0$ s.t. $R(t) \ge \delta \ \forall t, a \le t \le b$. If $\sigma(t), a \le t \le b$ is another path from z_0 to z_1 such that $|\sigma(t) - \gamma(t)| < \delta$ for $a \le t \le b$, then there is an analytic continuation $g_t(z)$ of $f_t(z)$ along σ and the terminal series $g_b(z)$ centered at $\sigma(b) = z_1$ coincides with $f_b(z)$

Monodromy Theorem: Let $f(z)$ be analytic at z_0. Let $\gamma_0(t)$ and $\gamma_1(t)$, $a \le t \le b$ be two paths from z_0 to z_1 along which $f(z)$ can be continued analytically. Suppose $\gamma_0(t)$ can be deformed continuously to $\gamma_1(t)$ by paths $\gamma_s(t)$ $0 \le s \le 1$ from z_0 to z_1 s.t. $f(z)$ can be continued analytically along each path γ_s. The analytic continuations of $f(z)$ along γ_1 and γ_0 coincide at z_1.



Use lemma

Chapter VI: Laurent Series and Isolated Singularities

1. The Laurent Decomposition

Splits function analytic on the annulus into a function analytic inside and one analytic outside the annulus

Theorem (Laurent Decomposition): Suppose $0 \le \rho < \sigma \le +\infty$ and suppose $f(z)$ is analytic for $\rho < |z - z_0| < \sigma$ then $f(z)$ can be decomposed

$$f(z) = f_0(z) + f_1(z)$$

$f_0(z)$ is analytic on $|z - z_0| < \sigma$ and $f_1(z)$ is analytic on $|z - z_0| > \rho$ and at ∞. If the decomposition is normalized so that $f_1(\infty) = 0$ then the decomposition is unique.

If $f(z)$ is analytic for $|z - z_0| < \sigma$ then the laurent decomposition is trivially $f_0(z) = f(z)$ and $f_1(z) = 0$.

If $f(z)$ is analytic for $|z-z_0| > \rho$ and $f(\infty)=0$, then the laurent decomposition is $f_0(z)=0$ and $f_1(z)=f(z)$

Uniqueness of decomposition follows from Liouville's theorem.

Let $f(z) = g_0(z) + g_1(z)$

$$f(z) = f_0(z) + f_1(z)$$

$$0 = f_0(z) - g_0(z) + f_1(z) - g_1(z)$$

$$h(z) = \begin{cases} f_0(z) - g_0(z) & |z-z_0| < \sigma \\ -(f_1(z) - g_1(z)) & |z-z_0| > \rho \end{cases}$$

Agree on overlap

$h(z)$ is analytic on all of $\mathbb{C}$

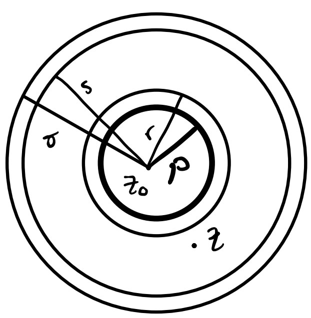
$\Downarrow$

bounded $\to$ Liouville $\to$ constant $\Rightarrow = 0$ $\quad f_0(z) = g_0(z)$ etc.

Finding decomposition:

Use cauchy integral formula, choose r, s such that $\rho < r < s < \sigma$

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta - z_0| = s} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \oint_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \text{for } r < |z-z_0| < s$$



Do we know that $f(z)$ is analytic for $z < \sigma$?

Is it even defined?

? $\quad f_0(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = s} \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad |z-z_0| < s$

$$f_1(z) = -\frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad |z - z_0| > r$$

? Both are analytic on their bounds

Uniqueness assertion removes need for r and s

Theorem: Laurent Series Expansion

Suppose $0 \le \rho < \sigma \le \infty$, and suppose $f(z)$ is analytic for $\rho < |z-z_0| < \sigma$. Then $f(z)$ has a Laurent expansion that converges absolutely at each point of the annulus and that converges uniformly on each subannulus $r \le |z-z_0| \le s$ where $\rho < r < s < \sigma$.

The coefficients are uniquely determined by $f(z)$ and they are given by $a_n = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz$, $-\infty < n < \infty$ for any fixed r, $\rho < r < \sigma$

Laurent Series expansion: $f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$ ← Obtained by summing power series of $f_0(z)$ and $f_1(z)$

Coefficients: Divide $f(z)$ in laurent series expansion by $(z-z_0)^{n+1}$ and integrate around $|z-z_0|=r$

$$\oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz = \oint \frac{1}{(z-z_0)^{n+1}} \sum_{k=-\infty} a_k (z-z_0)^k dz = \sum_{k=-\infty}^{\infty} a_k \oint_{|z-z_0|=r} (z-z_0)^{k-n-1} dz = 2\pi i a_n$$

$$\oint (z-z_0)^m = 2\pi i \text{ if } m=-1 \text{ and } 0 \text{ otherwise}$$

All terms but $2\pi i a_n$ in the series disappear

$$a_n = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz$$

The tail of the laurent series expansion with positive powers of $z-z_0$ converges on the largest open disk centered at z_0 to which $f_0(z)$ extends to be analytic

The tail of the series with negative powers of $z-z_0$ converges on the largest exterior domain of the form $|z-z_0| > \tau$ to which $f_1(z)$ extends analytically

Thus, the largest open domain on which the full laurent series converges is the largest open annular set centered at z_0 containing the annulus $\rho < |z-z_0| < \sigma$ to which $f(z)$ extends continuously

2. Isolated Singularities of an Analytic Function

A point z_0 is said to be an isolated singularity of $f(z)$ if $f(z)$ is analytic in some punctured disk centered at z_0

If $f(z)$ has an isolated singularity at z_0 then

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k, \quad 0 < |z-z_0| < r$$

Removable Singularity

Isolated singularity is said to be removable if $a_k = 0$ for all $k < 0$

Laurent series becomes a power series

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k \quad 0 < |z-z_0| < r$$

If $f(z_0) = a_0$ then the function is analytic on the entire disk

Riemann's Theorem on Removable Singularities: Let z_0 be an isolated singularity of $f(z)$. If $f(z)$ is bounded near z_0 then $f(z)$ has a removable singularity at z_0

Suppose $|f(z)| \leq M$ for z near z_0. Using the ML estimate of our coefficients

$$|a_n| \leq \frac{1}{2\pi} \frac{M}{r^{n+1}} (2\pi r) = \frac{M}{r^n} \qquad (r > 0 \text{ is small})$$

If $n < 0$ $\frac{M}{r^n} \to 0$ as $r \to 0$

$a_n = 0$ for $n < 0$ so the singularity is removable

Pole Singularity

Isolated singularity is a pole if there is $N > 0$ such that $a_{-N} \neq 0$ but $a_k = 0$ for all $k < -N$. Integer N is the order of that pole.

$$f(z) = \sum_{k=-N}^{\infty} a_k (z-z_0)^k = \frac{a_{-N}}{(z-z_0)^N} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

For negative powers

$P(z) = \sum_{k=-N}^{-1} a_k (z-z_0)^k$ is the principle part of $f(z)$ at the pole z_0. $P(z)$ accounts for bad behavior at z_0;

$f(z) - P(z)$ is analytic at z_0

Theorem: Let z_0 be an isolated singularity of $f(z)$. Then z_0 is a pole of order N if and only if $f(z) = \frac{g(z)}{(z-z_0)^N}$ where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$.

$\Rightarrow$ The power series $a_{-N} + a_{-N+1}(z-z_0) + a_{-N+2}(z-z_0)^2 + \ldots$ converges to $g(z)$ which is analytic at z_0

$$f(z) = \frac{g(z)}{(z-z_0)^N}$$

$\Leftarrow$ $f(z) = \frac{g(z)}{(z-z_0)^N}$ Power series representation of $\frac{g(z)}{(z-z_0)^N}$ is that of a pole of order N at z_0 for $f(z)$

Theorem: Let z_0 be an isolated singularity of $f(z)$. Then z_0 is a pole of $f(z)$ of order N if and only if $\frac{1}{f(z)}$ is analytic at z_0 and has a zero of order N

$\Rightarrow$ $f(z) = \sum_{-N}^{\infty} a_k (z-z_0)^k$

$g(z) = (z-z_0)^N f(z)$ -> analytic and $g(z_0) \neq 0$

$\frac{1}{g(z)}$ => analytic at z_0 and $h(z_0) \neq 0$.

$\frac{1}{f(z)} = \frac{(z-z_0)^N}{g(z)}$ => analytic at z_0 and has 0 of order N

<= reverse argument

$\frac{1}{f(z)}$ has 0 of order N => $\frac{1}{f(z)} = (z-z_0)^N h(z)$ where $h(z_0) = 0$ and $h(z)$ is analytic

$g(z) = \frac{1}{h(z)}$ so $f(z) = \frac{g(z)}{(z-z_0)^N}$ -> pole

A function is said to be meromorphic on a domain D if $f(z)$ is analytic on D except possibly at isolated singularities which are poles

Sums and products of meromorphic functions are meromorphic. Quotients are as well as long as the denominator isn't 0.

Theorem: Let z_0 be an isolated singularity of $f(z)$. Then z_0 is a pole if and only if $|f(z)| \to \infty$ as $z \to z_0$

=> If $f(z)$ has a pole of order N at z_0, $g(z) = (z-z_0)^N f(z)$ is analytic and non-zero at z_0

$$|f(z)| = \left| \frac{g(z)}{(z-z_0)^N} \right| \to \infty$$

<= Suppose $|f(z)| \to \infty$ as $z \to z_0$

$h(z) = \frac{1}{f(z)}$ is analytic in some punctured neighborhood near z_0.

$h(z) \to 0$ as $z \to z_0$

So $h(z)$ is analytic at z_0 and $h(z_0) =$

For order N zero of $h(z)$ $f(z)$ has a pole of order N at z_0

Essential Singularity

if $a_k \neq 0$ for infinitely many $k < 0$

Casorati-Weierstrass Theorem: Suppose z_0 is an essential isolated singularity of $f(z)$. Then for every complex number w_0 there is a sequence $z_n \to z_0$ such that $f(z_n) \to w_0$

Proof of contrapositive:

Suppose there exists a complex number w_0 that is not in the limit of values of $f(z)$ as above. Thus there exists a small $\varepsilon > 0$ s.t.

$|f(z) - w_0| > \varepsilon$ for all z near z_0.

$h(z) = \dfrac{1}{|f(z) - w_0|}$ is bounded near z_0 so from Riemann's theorem we know that z_0 is a removable singularity.

$h(z) = (z - z_0)^N g(z)$ for some $N \geq 0$ and some analytic $g(z)$ where $g(z) \neq 0$

Thus, $f(z) - w_0 = \dfrac{1}{h(z)} = \dfrac{1}{(z-z_0)^N} \cdot \dfrac{1}{g(z)}$

when $N = 0$ $f(z)$ extends analytically at z_0 but if $N > 0$ $f(z)$ has a pole of order N at z_0

3. Isolated Singularity at Infinity

$f(z)$ is said to have an isolated singularity at ∞ if $f(z)$ is analytic outside some bounded set.

If $R > 0$ s.t. $f(z)$ is analytic for $|z| > R$

$f(z)$ has isolated singularity at ∞ if and only if $g(z) = \dfrac{1}{f(z)}$ has isolated singularity at $z = 0$

$$f(z) = \sum_{k=-\infty}^{\infty} b_k z^k \qquad |z| > R$$

$f(z)$ at ∞ is removable if $b_k = 0$ for all $k > 0$ which means $f(z)$ is analytic at ∞

$f(z)$ at ∞ is essential if $b_k \neq 0$ for infinitely many $k > 0$

$f(z)$ has a pole of order N at ∞ if $b_N \neq 0$ while $b_k = 0$ for $k > N$

If $f(z)$ has a pole of order N at ∞

$$f(z) = \underbrace{b_N z^N + b_{N-1} z^{N-1} + \dots b_1 z + b_0}_{\text{principal part of } f(z)} + \frac{b_{-1}}{z}$$

4. Partial Fractions Decomposition

Theorem: A meromorphic function on the extended complex plane $\mathbb{C}^*$ is rational

A meromorphic function must have a finite number of poles

If $f(z)$ is analytic at ∞ let $P_\infty(z) = f(\infty)$

Otherwise $f(z)$ has a pole at $f(\infty)$ and $P_\infty(z)$ is the principal part of $f(z)$ at ∞

$f(z) - P_\infty(z) \to 0$ as $z \to \infty$

If $z_1, \ldots z_m$ are the poles of $f(z)$, let $P_k(z)$ be the principle part of $f(z)$ at z_k

$$P_k(z) = \frac{\alpha_1}{z - z_k} + \frac{\alpha_2}{(z-z_k)^2} + \cdots + \frac{\alpha_n}{(z-z_k)^n}$$

$P_k(z)$ is analytic at ∞ and vanishes there

If $g(z) = f(z) - P_\infty(z) - \sum_{j=1}^{m} P_j(z)$, $g(z)$ is an entire function since each portion is analytic at z_k

$g(z) \to 0$ as $z \to \infty$ and by Liouville's theorem $\Rightarrow g(z) = 0$

$$f(z) = \underbrace{P_\infty(z) - \sum_{j=1}^{m} P_j(z)}_{\text{Partial fraction Decomposition}}$$

f is rational

Theorem: Every rational function has a partial fractions decomposition expressing it as a sum of polynomial in z and its principal parts at each of its poles in the complex plane

Division Algorithm

For arbitrary polynomials $p(z)$ and $q(z)$

$$p(z) = c_0 z^{n-m} q(z) + p_1(z) = c_0 z^{n-m} q(z) + c_1 z^{n_1 - m} q(z) + \cdots + c_{k-1} z^{n_{k-1}-m} q(z) + P_k(z)$$

Chapter VII : The Residue Calculus

1. The Residue Theorem

For a given isolated singularity z_0 of $f(z)$

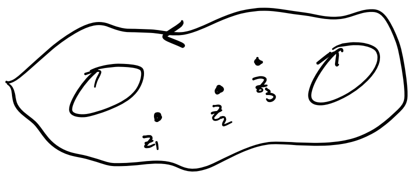
$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \qquad 0 < |z - z_0| < \rho$$

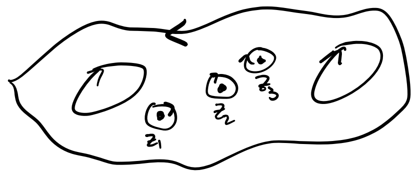
The residue of $f(z)$ at z_0 is a_{-1} of $\frac{1}{z-z_0}$

$$\operatorname{Res}[f(z), z_0] = a_{-1} = \frac{1}{2\pi i} \oint_{|z-z_0|=r} f(z)\,dz \qquad 0 < r < \rho$$

Residue Theorem: Let D be a bounded domain in the complex plane with piecewise smooth boundary. Suppose that $f(z)$ is analytic on $D \cup \partial D$ except for a finite number of isolated singularities $z_1, \dots z_m$ in D.

$$\int_{\partial D} f(z)\,dz = 2\pi i \sum_{j=1}^{m} \text{Res}\left[f(z), z_j\right]$$





Let D_ε be the domain obtained from D by punching out small disks U_j centered around z_j with radius ε.

$$\int_{\partial U_j} f(z)\,dz = 2\pi i \,\text{Res}\left[f(z), z_j\right]$$ ← from definition of Residue

Cauchy Theorem

$$0 = \int_{\partial D_\varepsilon} f(z)\,dz = \int_{\partial D} f(z)\,dz - \sum_{j=1}^{m} \int_{\partial U_j} f(z)\,dz$$

$$\Rightarrow \int_{\partial D} f(z)\,dz = 2\pi i \sum_{j=1}^{m} \text{Res}\left[f(z), z_j\right]$$

Rule 1: If $f(z)$ has a simple pole at z_0

$$\text{Res}\left[f(z), z_0\right] = \lim_{z \to z_0} (z - z_0) f(z)$$

Laurent series for $f(z)$

$$f(z) = \frac{a_{-1}}{z - z_0} + [\text{analytic at } z_0]$$

Rule 2: If $f(z)$ has a double pole at z_0, then

$$\text{Res}\left[f(z), z_0\right] = \lim_{z \to z_0} \frac{d}{dz}\left[(z - z_0)^2 f(z)\right]$$

$$f(z) = \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + \cdots$$

$$(z-z_0)^2 f(z) = a_{-2} + a_{-1}(z-z_0) + a_0(z-z_0)^2$$

Rule 3: If $f(z)$ and $g(z)$ are analytic at z_0 and if $g(z)$ has a simple 0 at z_0 then

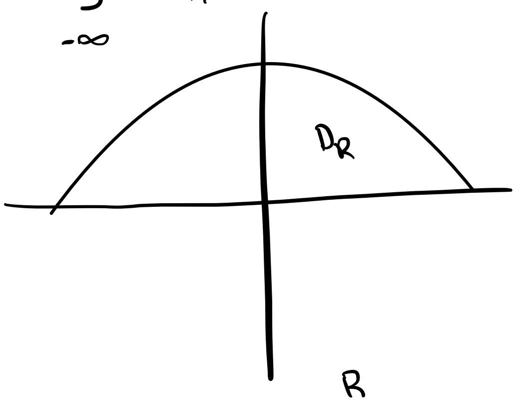
$$\text{Res}\left[\frac{f(z)}{g(z)}, z_0\right] = \frac{f(z_0)}{g'(z_0)}$$

Rule 4: If $g(z)$ is analytic and has a simple zero at z_0, then

$$\text{Res}\left[\frac{1}{g(z)}, z_0\right] = \frac{1}{g'(z_0)}$$

2. Integrals Featuring Rational Functions

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$



$\frac{1}{1+z^2}$ has one pole in D_R at i with residue $\frac{1}{2i}$

$$\int_{\partial D_R} \frac{dz}{1+z^2} = 2\pi i \text{ Res}\left[\frac{1}{1+z^2}, i\right] = 2\pi i \cdot \frac{1}{2i} = \pi$$

$$\int_{\partial D_R} \frac{dz}{1+z^2} = \int_{-R}^{R} \frac{dx}{1+x^2} + \int_{T_R} \frac{dz}{1+z^2}$$

Length of T_R is πR so using ML-estimates

$$\left| \int_{T_R} \frac{dz}{1+z^2} \right| \leq \frac{1}{R^2-1} \cdot \pi R \sim R$$

As $R \to \infty$ $\int_{-R}^{R} \frac{dx}{1+x^2} = \pi$ since $\int_{T_R} \frac{dz}{1+z^2} \to 0$

3. Integrals of Trigonometric Functions

Consider $\int_0^{2\pi} \frac{d\theta}{a+\cos\theta}$ $a>1$

Let $z = e^{i\theta}$ so $dz = ie^{i\theta}$ to parameterize around the unit circle

$$d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+1/z}{2}$$

$$\int_0^{2\pi} \frac{d\theta}{a+\cos\theta}\, d\theta = \int_{|z|=1} \frac{1}{a+\frac{1}{2}(z+1/z)} \frac{dz}{iz} = \frac{2}{i}\int_{|z|=1} \frac{dz}{z^2+2az+1}$$

poles are at zeros of $z^2+2az+1$

Only one root is in the unit circle: $z_0 = -a+\sqrt{a^2-1}$

$$\text{Res}\left[\frac{1}{z^2+2az+1}, z_0\right] = \frac{1}{2z+2a}\Bigg|_{z=z_0} = \frac{1}{2\sqrt{a^2-1}}$$

$$\int_0^{2\pi} \frac{d\theta}{a+\cos\theta} = \frac{2}{i}\cdot 2\pi i \cdot \frac{1}{2\sqrt{a^2-1}} = \frac{2\pi}{\sqrt{a^2-1}}$$

4. Integrands with Branch Points

Identity used for integrands of x^a and $\log x$ using contour integration

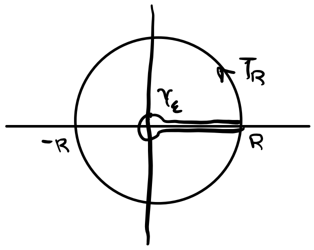
$$\int_0^{\infty} \frac{x^a}{(1+x)^2}\, dx = \frac{\pi a}{\sin(\pi a)} \qquad -1<a<1$$

Consider the branch of the function $\frac{z^a}{(1+z)^2}$ on the slit plane $\mathbb{C}\setminus[0,+\infty]$

$$f(z) = \frac{r^{\alpha} e^{i\alpha\theta}}{(1+z)^2} \qquad z = re^{i\theta} \qquad 0<\theta<2\pi$$

Using continuity we can extend $f(z)$ to the upper and lower portions of the slit

$\theta = 0$ at the top edge and $\theta = 2\pi$ on the bottom edge



For small $\varepsilon > 0$ and large $R > 0$ we can consider the keyhole domain D

$f(z)$ has a double pole at $z = -1$

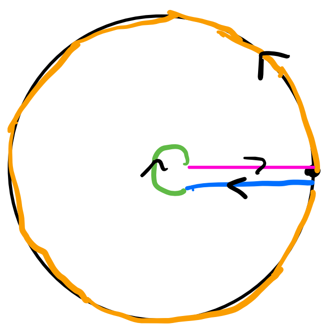
Using Residue Rule 2

$$\text{Res}\left[\frac{z^a}{(1+z)^2}, -1\right] = \frac{d}{dz} z^a \Big|_{z=-1} = a\frac{z^a}{z}\Big|_{z=-1} = -ae^{\pi i a}$$

Residue Theorem:

$$\int_{\partial D} f(z)dz = -2\pi i a e^{\pi i a}$$

Integral can be split into four separate integrals



$$= \int_\varepsilon^R \frac{x^a}{(1+x)^2}dx + \int_{T_R} f(z)dz + \int_R^\varepsilon \frac{e^{2\pi i a}x^a}{(1+x)^2} + \int_{\gamma_\varepsilon} f(z)dz$$

Using ML-estimates:

$$\left|\int_{T_R} \frac{z^a}{(1+z)^2}dz\right| \leq \frac{R^a}{(R-1)^2} \cdot 2\pi R \sim R^{a-1}$$

$$\left|\int_{\gamma_\varepsilon} \frac{z^a}{(1+z)^2}dz\right| \leq \frac{\varepsilon^a}{(1-\varepsilon)^2} \cdot 2\pi\varepsilon \sim \varepsilon^{a+1}$$

As $R \to \infty$ and $\theta \to 0$ these integrals approach 0

Reversing direction of integration from $R+\varepsilon$ we find

$$-2\pi i a e^{\pi i a} = (1-e^{2\pi i a})\int_0^\infty \frac{x^a}{(1+x)^2}dx$$

$$\int_0^\infty \frac{x^a}{(1+x)^2}dx = \frac{-2\pi i a e^{\pi i a}}{1-e^{2\pi i a}} = \frac{2\pi i a}{e^{\pi i a}-e^{-\pi i a}} = \frac{\pi a}{\sin(\pi a)}$$

5. Fractional Residues

Suppose z_0 is an isolated singularity of $f(z)$. For $\varepsilon > 0$ small consider

$$\int_{C_\varepsilon} f(z)dz \qquad C_\varepsilon \text{ is an arc of } \{|z-z_0|<\varepsilon\}$$

Fractional Residue Theorem: If z_0 is a simple pole of $f(z)$, and C_ε is an arc of the circle $|z-z_0|<\varepsilon$ with angle α, then

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon} f(z)dz = \alpha i \operatorname{Res}[f(z), z_0]$$

Proof: $f(z) = \dfrac{A}{(z-z_0)} + g(z)$ where A is $\operatorname{Res}[f(z), z_0]$ and $g(z)$ is analytic at z_0.

If we parameterize the circle as $z = z_0 + \varepsilon e^{i\theta}$

$$\int_{C_\varepsilon} \frac{A}{z-z_0}dz = iA\int_{\theta_0}^{\theta_0+\alpha} d\theta = \alpha i A$$

Since $g(z)$ is bounded near z_0 and the length of C_ε is at most $2\pi\varepsilon$ our ML estimates tell us $\int_{C_\varepsilon} g(z)dz \to 0$ as $\varepsilon \to 0$.

Therefore, as $\varepsilon \to 0$ $\int_{C_\varepsilon} f(z)dz \to i\pi a_{-1}$

6. Principal Values

An integral $\int_a^b f(x)dx$ is said to be absolutely convergent if $\int_a^b |f(x)|dx$ is finite

" absolutely divergent if $\int_a^b |f(x)|\,dx = \infty$

Principal Value of an Integral

$$PV \int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \left(\int_a^{x_0-\varepsilon} + \int_{x_0+\varepsilon}^b \right) f(x)\,dx$$

Principal value coincides with integral if $f(x)$ is absolutely integrable

If $f(x)$ has a finite number of discontinuities PV of $f(x)$ can be calculated by dividing (a,b) into subintervals containing one discontinuity of $f(x)$ and summing the PV of each sub interval

See page 213 for example

Hilbert Transform

$$(Hu)(t) = PV \int_{-\infty}^{\infty} \frac{u(s)}{s-t}\,ds \qquad -\infty < t < \infty$$

$u(s)$ is an integrable function on the real line

7. Jordan's Lemma

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin x\,dx, \quad \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos x\,dx \qquad \deg Q(x) = \deg P(x) + 1$$

Absolutely divergent

Jordan's Lemma: If Γ_R is the semicircle contour $z(\theta) = Re^{i\theta}$, $0 \le \theta \le \pi$ in the upper half plane

then $\int_{\Gamma_R} |e^{iz}|\,|dz| < \pi$

For $z(\theta) = Re^{i\theta}$ $\quad |e^{iz}| = e^{-R\sin\theta} \quad |dz| = R\,d\theta$

$$\int_0^{\pi} e^{-R\sin\theta}\,d\theta < \frac{\pi}{R}$$

Since $\sin\theta \geq 2\theta/\pi \quad 0 \leq \theta \leq \pi/2$

$$\int_0^{\pi} e^{-R\sin\theta}\,d\theta = 2\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta \leq 2\int_0^{\pi/2} e^{-2R\theta/\pi}\,d\theta = \frac{\pi}{R}\int_0^{R} e^{-t}\,dt < \frac{\pi}{R}\int_0^{\infty} e^{-t}\,dt = \frac{\pi}{R}$$

8. Exterior Domains

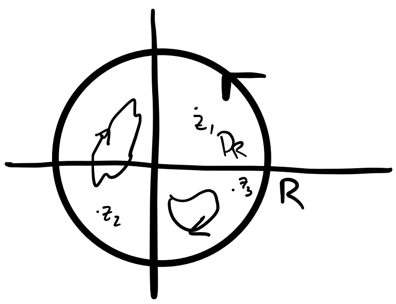
Exterior domain is a Domain D in the complex plane that includes all z such that $|z| \geq R$ for some R

Residue Theorem must be modified for ∞

Theorem: Let D be an exterior domain with piecewise smooth boundary. Suppose $f(z)$ is analytic on $D \cup \partial D$ except for a finite number of singularities $z_1, \ldots z_m$ in D and let a_{-1} be the coefficient of $\frac{1}{z}$ in the laurent expansion of $f(z)$

$$\int_{\partial D} f(z)dz = -2\pi i a_{-1} + 2\pi i \sum_{j=1}^{m} \text{Res}\left[f(z), z_j\right]$$

Proof: Consider



$$\int_{\partial D} f(z)dz + \int_{|z|=R} f(z)dz = \int_{\partial D_R} f(z)dz \overset{\text{residue thm}}{=} 2\pi i \sum_{j=1}^{m} \text{Res}\left[f(z), z_j\right]$$

D_R is $z \in D$ s.t. $|z| < R$

Substituting the laurent series expansion of $f(z)$ into $\int_{|z|=R} f(z)dz$ gets us $2\pi i a_{-1}$

Integral formula for a_{-1} coefficient

$$\int_{\partial D} f(z)\,dz = \int_{\partial D_R} f(z)dz - \int_{|z|=R} f(z)dz = 2\pi i \sum_{j=1}^{m} \text{Res}\left[f(z), z_j\right] - 2\pi i a_{-1}$$

Suppose $f(z)$ is analytic for $|z| \geq R$ with laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \quad |z| \geq R$

$$\text{Res}\left[f(z), \infty\right] = -a_{-1}$$

Essentially the residue theorem for bounded domains is identical to exterior domains except for the inclusion of the residue at ∞

Chapter VIII : The Logarithmic Integral

1. The Argument Principal

Suppose $f(z)$ is analytic on a domain D. For a curve γ in D such that $f(z) \neq 0$ on γ

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_\gamma d\log f(z)$$

is the logarithmic integral of $f(z)$ along γ.

Theorem: Let D be a bounded domain with piecewise smooth boundary ∂D and let $f(z)$ be meromorphic function on D that extends to be analytic on ∂D, such that $f(z) \neq 0$ on ∂D.

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)}\,dz = N_0 - N_\infty$$

where N_0 is the number of zeros of $f(z)$ in D and N_∞ is the number of poles of $f(z)$ in D, counting multiplicities

$N_0 - N_\infty$ is the residue theorem for $\frac{f'(z)}{f(z)}$

Suppose we have a pole of order N at z_0 of $f(z)$

$$f(z) = (z-z_0)^N g(z)$$

$$\frac{f'(z)}{f(z)} = \frac{N(z-z_0)^{N-1}\,g(z)}{(z-z_0)^N\,g(z)} + \frac{(z-z_0)^N g'(z)}{(z-z_0)^N g(z)} = \frac{N}{z-z_0} + \text{analytic}$$

$\frac{f'(z)}{f(z)}$ then has a simple pole at z_0 with residue N

For each zeros and poles of $\frac{f'(z)}{f(z)}$ in D we find $N_0 - N_\infty$

$$\frac{1}{2\pi i}\int_\gamma d\log f(z) = \frac{1}{2\pi i}\int_\gamma d\log|f(z)| + \frac{1}{2\pi i}\int_\gamma d\arg(f(z))$$

$d\log(f(z))$ is exact

Parameterizing $\gamma(t) = x(t) + iy(t)$, $a \le t \le b$

$$\int_\gamma d\log|f(z)| = \log|f(\gamma(b))| - \log|f(\gamma(a))|$$

$$\int_\gamma d\log|f(z)| = 0 \quad \text{when } \gamma \text{ is a closed curve}$$

$$\int_\gamma d\arg(f(z)) = \arg f(\gamma(b)) - \arg f(\gamma(a))$$ ← increase in argument

Theorem: Let D be a bounded domain with piecewise smooth boundary ∂D. $f(z)$ is meromorphic on D that extends to be analytic on ∂D, s.t. $f(z) \neq 0$ on ∂D. Then

$$\int_{\partial D} d\arg(f(z)) = 2\pi(N_0 - N_\infty)$$

2. Rouché's Theorem

Small perturbations don't change the number of 0's of an analytic function

Rouché's Theorem: Let D be a bounded domain with a piecewise smooth boundary ∂D. Let $f(z)$ and $h(z)$ be analytic on $D \cup \partial D$. If $|h(z)| < |f(z)|$ for $z \in \partial D$, then $f(z)$ and $f(z)+h(z)$ have the same number of zeros in D, counting multiplicity

$$f(z)+h(z) = f(z)\left[1 + \frac{h(z)}{f(z)}\right]$$

Since $\left|\frac{h(z)}{f(z)}\right| < 1$, $1+\frac{h(z)}{f(z)}$ are in the right half of the plane

So increase of $\arg\left(1+\frac{h(z)}{f(z)}\right) = 0$ for a closed boundary

So, increase of $\arg f(z) = \underset{\text{increase}}{\arg} f(z)+h(z)$

Finally, applying argument principle we get that the they have the same zeros

3. Hurwitz's Theorem

Hurwitz's Theorem: Suppose $\{f_k(z)\}$ is a sequence of analytic functions on a domain D that converges normally on D to $f(z)$, and suppose $f(z)$ has a zero of order N at z_0. Then there exists $\rho > 0$ such that for large k, $f_k(z)$ has exactly N zeros in the disk $\{|z-z_0| < \rho\}$, counting multiplicity and these zeros converge to z_0 as $k \to \infty$

Let $\rho > 0$ be sufficiently small so that the closed disk $\{|z-z_0| \le \rho\}$ is contained D so $f(z) \neq 0$ in the disk $0 < |z-z_0| \le \rho$. Choose $\delta > 0$ such that $|f(z)| \ge \delta$ on the circle $|z-z_0| = \rho$

We know that $f_k(z)$ converges uniformly to $f(z)$, so for large k we have $|f_k(z)| > \delta/2$ for $|z-z_0| \le \rho$

and $\frac{f_k'(z)}{f_k(z)}$ converges uniformly to $\frac{f'(z)}{f(z)}$.

$$\text{Thus, } \underbrace{\frac{1}{2\pi i}\int_{|z-z_0|=\rho} \frac{f_k'(z)}{f_k(z)}\,dz}_{\substack{\text{Number of } N_k \text{ zeros of } f_k(z) \\ \text{in the disk } \{|z-z_0|<\rho\}}} \longrightarrow \underbrace{\frac{1}{2\pi i}\int_{|z-z_0|=\rho} \frac{f'(z)}{f(z)}\,dz}_{\substack{\text{Number of zeros of } f(z) \\ \text{on the disk}}}$$

A function is univalent on a domain D if it is analytic and one to one on D

Theorem: Suppose $\{f_k(z)\}$ is a sequence of univalent function on a domain D that converges normally on D to a function $f(z)$. Then either $f(z)$ is univalent or $f(z)$ is constant

Proof by contradiction with $f(z_0) = f(\zeta_0) = w_0$ and the function $f(z) - w_0$.

4. Open Mapping and Inverse Function Theorem

Let $f(z)$ be a meromorphic function on a domain D. $f(z)$ attains the value w_0 m times at z_0 if $f(z)-w_0$ has a zero of order m at z_0

Let $f(z)$ be a nonconstant analytic function on a domain D. Let $z_0 \in D$, $w_0 = f(z_0)$ and assume $f(z)-w_0$ has a zero of order m at z_0

Since the zeros are isolated we can select $\rho > 0$ where $f(z) - w_0 \neq 0$ for $0 < |z-z_0| \le \rho$

Let δ be the minimum of $|f(z) - w_0|$ on $|z-z_0| = \rho$

$$N(w) = \frac{1}{2\pi i}\int_{|z-z_0|=\rho} \frac{f'(z)}{f(z)-w}\,dz \qquad |w-w_0| < \delta$$

$N(w)$ is the number of zeros of $f(z)-w$ in the disk $\{|z-z_0|<\rho\}$ and must be integer valued

This implies that $N(w)$ is constant and since we know $N(w_0) = m$, $N(w) = m$

Open Mapping Theorem for Analytic Functions: If $f(z)$ is analytic on a domain D, and $f(z)$ is not constant, then $f(z)$ maps open sets to open sets, that is, $f(U)$ is open for each open subset U of D.

Direct application of proof above

Inverse Function Theorem: Suppose $f(z)$ is analytic for $|z-z_0| \le \rho$ and satisfies $f(z_0)=w_0$, $f'(z_0) \neq 0$ and $f(z) \neq w_0$ for $0 < |z-z_0| \le \rho$. Let $\delta > 0$ be chosen such that $|f(z)-w_0| \ge \delta$ for $|z-z_0| = \rho$. Then for each w such that $|w-w_0| < \delta$ there is a unique z satisfying $|z-z_0|<\rho$ and $f(z)=w$. $z = f^{-1}(w)$

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = \rho} \frac{\zeta f'(\zeta)}{f(\zeta)-w} \, d\zeta \qquad |w-w_0| < \delta$$

$$\text{Res}\left[\frac{\zeta f'(\zeta)}{f(\zeta)-w}, z \right] = \lim_{\zeta \to z} \frac{(\zeta - z)\, \zeta f'(z)}{f(\zeta)-w} = z$$

6. Winding Numbers

For a closed path $\gamma(t)$, $a \le t \le b$ in D the trace of γ is defined as $\Gamma = \gamma([a,b])$

For $z_0 \notin \Gamma$ the winding number:

$$W(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-z_0} = \frac{1}{2\pi} \int_\gamma d \arg(z-z_0) \qquad z_0 \notin \Gamma$$

Theorem: Let $\gamma(t)$, $a \le t \le b$, be a closed path in the complex plane and let $\Gamma = \gamma([a,b])$ be its trace. $W(\gamma, \zeta)$ is constant on each connected component of $\mathbb{C} \setminus \Gamma$. $W(\gamma, \zeta) = 0$ for all ζ in the unbounded component of $\mathbb{C} \setminus \Gamma$

Theorem: If $f(z)$ is analytic on domain D, then $\int_\gamma f(z)dz = 0$ for each closed path γ in D such that $W(\gamma, \zeta) = 0$ for all $\zeta \in \mathbb{C} \setminus D$

Proof on page 243

Theorem: Let $f(z)$ be analytic on domain D, and let γ be a closed path in D with $\Gamma = \gamma(a,b)$ if $W(\gamma, \zeta) = 0$ for all $\zeta = \mathbb{C} \setminus D$

$$W(\gamma, z_0) f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-z_0} \, dz$$

8. Simply Connected Domains

Simply connected if it has "no holes"

Let $\gamma(t)$, $a \le t \le b$ be a closed path in a domain D. γ is deformable to a point if there are closed paths $\gamma_s(t)$, $a \le t \le b$ and $0 \le s \le 1$ in D s.t. $\gamma_s(t)$ depends continuously on s and t, $\gamma_0 = \gamma$ and $\gamma_1(t) \equiv z_1$.

A domain is simply connected if every closed path in D can be deformed to a point

Lemma: Let $\gamma(t)$, $0 \le t \le 1$ be a closed path in D, with $z_0 = \gamma(0) = \gamma(1)$. Suppose that γ can be deformed continuously to a point in D. Then there is a continuous family of closed paths γ_s $0 \le s \le 1$ such that $\gamma_0 = \gamma$, γ_1 is the constant path at z_0 and each γ_s starts and ends at z_0

Theorem: For a domain D in the complex plane, the following are equivalent

i) D is simply connected

ii) Every closed differential on D is exact

iii) For each $z_0 \in \mathbb{C} \setminus D$ there is an analytic branch of $\log(z - z_0)$ defined on D

iv) each closed curve in D has winding number $W(\gamma, z_0) = 0$ about all points $z_0 \in \mathbb{C} \setminus D$

v) The complement of D in the extended complex plane is connected

Proofs on page 255

Chapter 9: The Schwarz Lemma and Hyperbolic Geometry

1. The Schwarz Lemma

Schwarz Lemma: Let $f(z)$ be analytic for $|z| < 1$. Suppose $|f(z)| \le 1$ for all $|z| < 1$, and $f(0) = 0$. Then

$$|f(z)| \le |z| \qquad |z| < 1$$

and if the equality holds at any point other than $z_0 = 0$, then $f(z) = \lambda z$ for some constant λ unit of modulus

Consider $f(z) = z g(z)$ where $g(z)$ is analytic

Let $r < 1$ and $|z| = r$

$$|g(z)| = \frac{|f(z)|}{r} \le \frac{1}{r}$$

Via maximum principle this holds for all $|z| \leq r$

As $r \to 1$ $|g(z)| \leq 1$

If $|f(z_0)| = |z_0|$, $|g(z)| = 1$ and by strict maximum principle $g(z)$ is constant, $g(z) = \lambda$

Theorem: Let $f(z)$ be analytic for $|z| < 1$. If $|f(z)| \leq 1$ for $|z| < 1$ and $f(0) = 0$, then

$$|f'(0)| \leq 1$$

with equality iff $f(z) = \lambda z$ where $|\lambda| = 1$

Consider derivative as $z \to 0$ $g(0) = f'(0)$